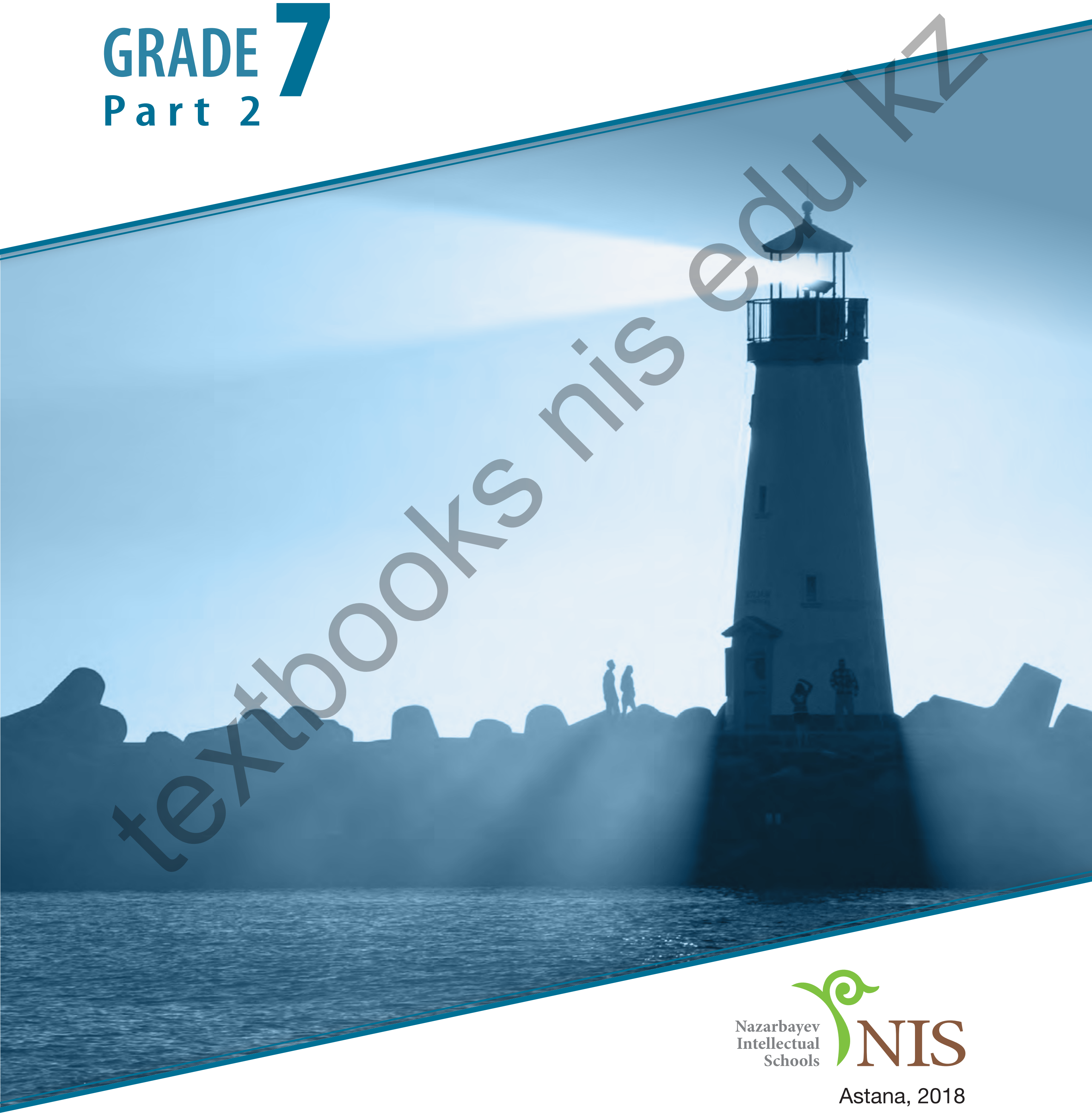
Maths

GRADE 7
Part 2



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Introduction

Dear friends!

It is an extraordinary textbook you are holding. It will guide you to an existing and diverse world of mathematics.

In this course, you will: extend your knowledge of algebra, geometry and statistics; learn how to describe various situations in mathematical language; make mathematical models; and other new things. However, the most significant skill for you to acquire is the ability to learn independently. You will learn how to set goals, make action plans and evaluate your working.

Self-assessment pages at the end of each unit provide you with an opportunity to

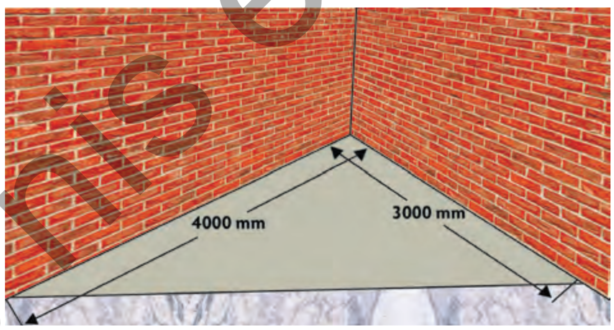
track your progress. This will help you become more confident and successful both in mathematics and in its application in everyday life.

Authors

1 Triangles

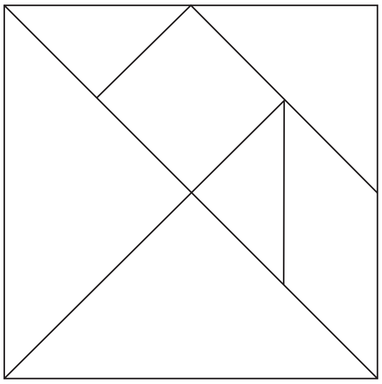
By the end of this unit, I will have learned:
- ✔ what triangle is;
- ✔ types of triangles;
- ✔ what is median, bisector and altitude of triangle;

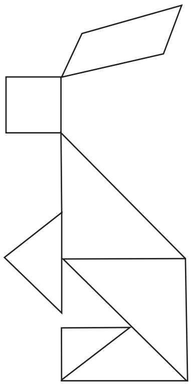
I will be able to:
- ✔ prove congruence of triangles;
- ✔ Solve problems using conditions for congruent triangles.



Triangle is the basic geometric figure. Properties of triangles are related to problem solving in architecture, astronomy, navigation and geometry itself. Did you know that any polygon can be divided on finite number of triangles.

Tangram is ancient eastern puzzle, consisting of figures, which were obtained when square was cut on 7 parts.

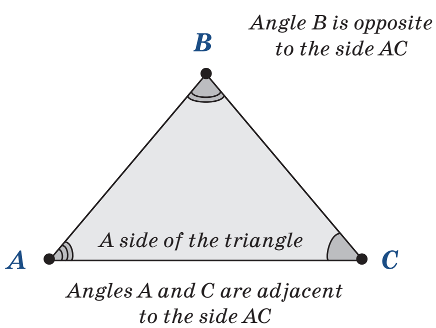




1.1 Triangle and its types

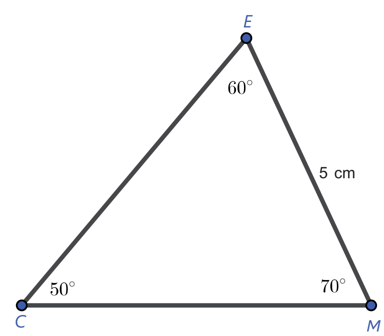
1. **Draw following the plan.**
   a) Mark three non-collinear points on the sheet of paper. Name them A, B and C.
   b) Connect the points. What segments did you get?
   c) Shade the interior of the resulting figure.
   What figure did you get?

> **REMEMBER!**
>
> Triangle is a part of a plane limited by three non-collinear points and by three segments that connect these points.
>
> The letters of the vertices written in capital Latin, *ABC*, name a triangle.



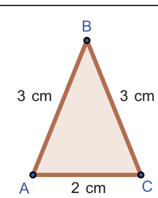
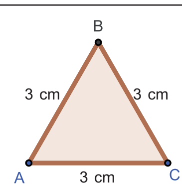
Points A, B, C – are vertices of the triangle.
Segments AB, BC, AC are sides of the triangle.
$\angle A$, $\angle B$, $\angle C$ — are angles of the triangle.

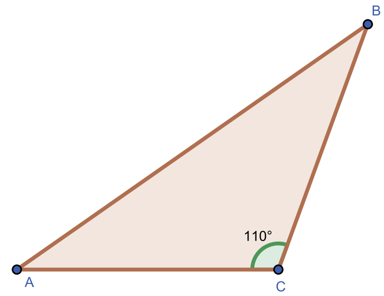
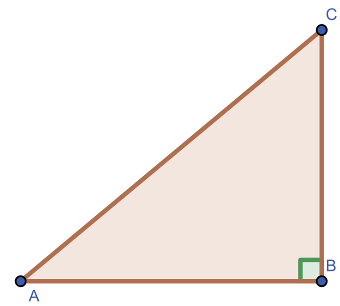
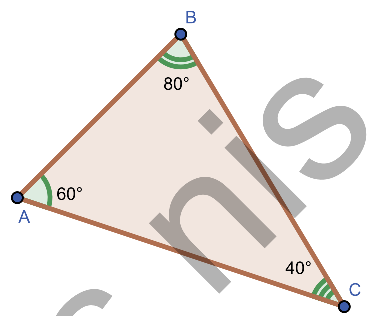
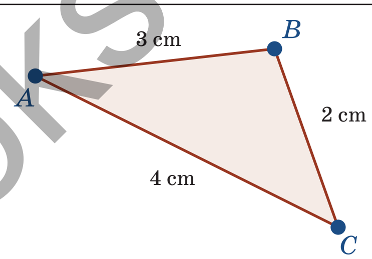
2. **Complete the task using the drawing.**
   a) Write down all possible names of the triangle.
   b) Show:
      - a side opposite to the angle C;
      - an angle opposite to the side CM;
      - angles adjacent to the sides EC and EM.
   c) Write down the name of the smallest side and the largest angle of the triangle.



Revise your previous knowledge on triangles using the task below.

3. **Match the type, drawings and definitions of triangles.**

	Type		Drawing		Definition
A.	Equilateral triangle	1.		I.	Acute triangle is a triangle with three acute angles.
B.	Isosceles triangle	2.		II.	Obtuse triangle is a triangle with one obtuse angle.

	Type		Drawing		Definition
C.	Right triangle	3.		III.	Right triangle is a triangle with one right angle.
D.	Acute triangle	4.		IV.	Isosceles triangle is a triangle with two equal sides.
E.	Obtuse triangle	5.		V.	Equilateral triangle is triangle with three equal sides
F.	Scalene triangle	6.		VI.	Scalene triangle is a triangle with three side of different length

4. Draw a triangle using a ruler and a protractor:

a) triangle MNK with $MN = 5$ cm, $KN = 5$ cm, and $\angle N = 110^{o}$;

b) triangle PQR ($\angle PQR = 90^{o}$), with sides PQ and QR equal to 3 cm and 4 cm respectively.

Define the type of each triangle.

5. Are the following statements correct? Explain why.

a) Triangle is isosceles if its two sides are equal.

b) Obtuse triangle can be isosceles.

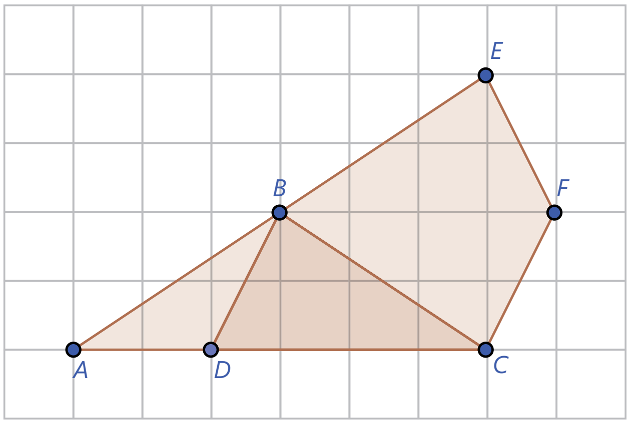
c) Triangle with two obtuse angles exists.

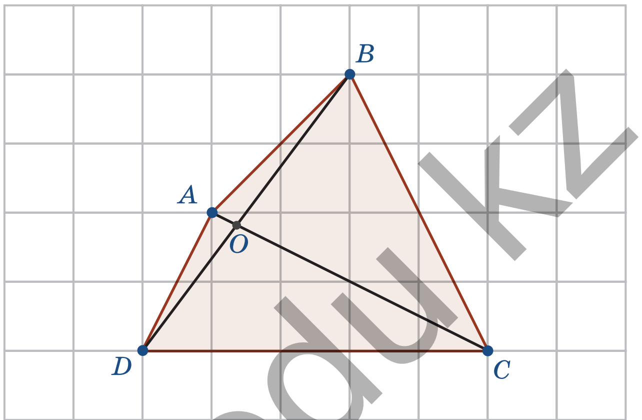
d) Equilateral triangle can be isosceles.

e) Obtuse triangle can be right.

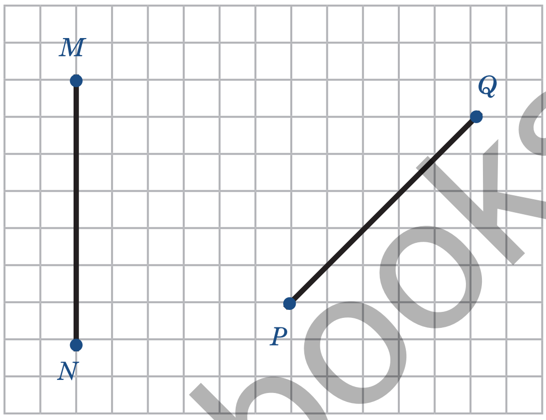
1.2 Problem solving

1. **How many triangles are on the drawings below? Name their elements and define their types..**





2. **Maya has drawn isosceles triangles MNK ($MN = MK$) and PQR ($PQ = QR$) with lateral sides MN and PQ. However, some parts of the drawing have been wiped out. Restore the original drawing.**



What triangles will Maya get if $MK = KN$ with angle PQR equals 90^o?

REMEMBER!

Perimeter of a triangle equals the sum.

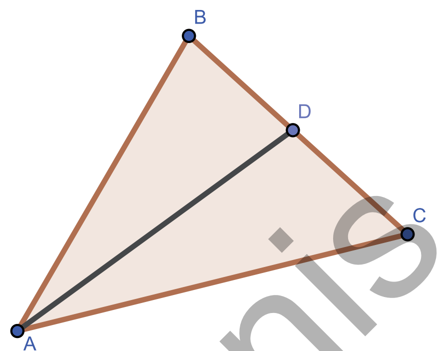
3. **Fill in the table.**

Type of triangle	Side AB (cm)	Side BC (cm)	Side AC (cm)	Perimeter of triangle
	28	46	51	
Isosceles ($AB = BC$)	2,5		2,6	
	16	16	16	
	18	18	32	
Right	10	24	26	

4. **Solve the following problems.**
   a) PQR is a triangle with a side PQ equals 10 cm, side QR is 1.5 times the side PQ, and side PR is by 3 less then side QR. What is the perimeter of the triangle PQR?
   b) ABC is an isosceles triangle ($AB = BC$) with the base is ¼ of the lateral side and a perimeter equals to 45 см. Find the length of three sides.
   c) Sum of two sides of isosceles triangle equals 26 cm and a perimeter equals 36 cm. What can you tell about the sides of the triangle?

5. **Arman claims that he needs only six matches to make a figure that consists of four equilateral triangles with a side equals to the length of a match. Is he right? Can you recreate his solution?**

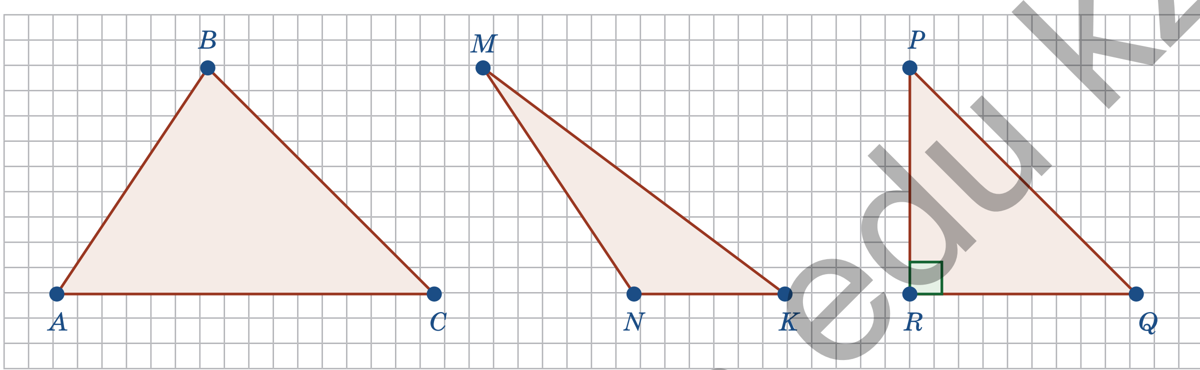
6. **How many triangles are there on the picture below? Can you draw two lines so that the total number of triangles are 5? 6 triangles? Explain why.**



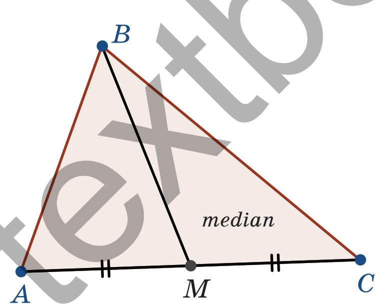
1.3 Median, bisector and altitude of a triangle

Triangle is an essential figure of plane geometry. We can draw all sorts of lines and segments in triangle that have special names and properties. In this lesson, we going to talk more about them.

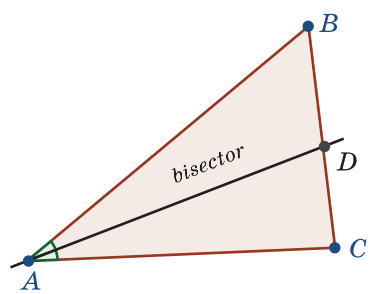
1. Draw following the plan.
Copy the triangles below to your copybook.



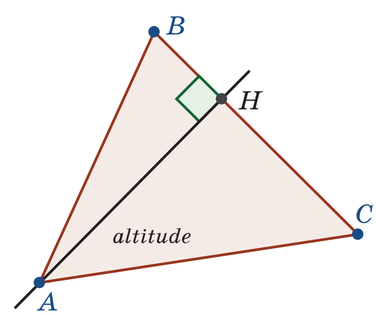
a) Mark midpoints of BC, MN and RQ and name them D, L and T respectively. Connect the resulting points with opposite vertices. The resulting segments are called **medians** of the triangles.
b) Use protractor to mark angle bisectors B, N and Q of the triangles. For each triangle mark a point where angle bisector intersects with the side of the triangle. What segments of angle bisectors did you get? These segments are called **bisectors** of the triangles.
c) Use set square to draw perpendicular lines from vertices C, M and R. For each triangle mark a point where the perpendicular line intersects with the side of the triangle. What segments of perpendicular lines did you get? These segments are called **altitudes** of the triangles.



Segment BM of the line, connecting the vertex



Segment AD of bisector of the triangle



Segment AH of a perpendicular line

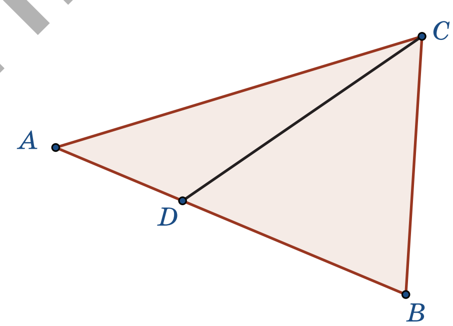
of the triangle with the midpoint of the opposite side is called median of triangle.	drawn from vertex to intersecting point with the opposite side of a triangle is bisector of a triangle.	drawn from vertex of a triangle to the opposite side, is called altitude of triangle.
ΔABC $AM = MC$,	ΔABC $\angle BAD = \angle DAC$	ΔABC $AH \perp BC$
Segment BM — is median of the triangle ABC	Segment AD is bisector of the triangle ABC	Segment AH is altitude of the triangle ABC

2. Draw scalene obtuse triangle and draw median, bisector and altitude of a triangle from vertex of larger angle. What can you tell about relative position of these segments?

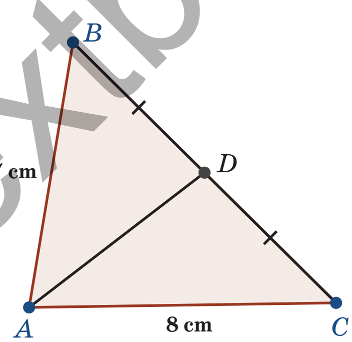
3. Are the statements correct? Show your solution in drawing.
   a) 4 medians can be drawn in any triangle.
   b) 3 altitudes can be drawn in any triangle.
   c) All medians of a triangle are inside the triangle.
   d) Altitude of a triangle can be outside the triangle.
   e) Bisectors of a triangle can be outside a triangle.
   f) A triangle with altitude coinsiding with one of its sides exists.

4. Median *BM* divides triangle *ABC* on two triangles, which perimeters are 34 cm and 36 cm respectively. What is perimeter of the triangle *ABC*, if *BM* =8cm?

5. How many triangles can you see on the drawing? Draw altitude of a triangle, which will be common for all triangles.



6. Solve a problem.



Given: □ABC
AD — median,
$AB = 7$ cm,
$AC = 8$ cm.

Find: $P_{\Delta ACD}$, $P_{\Delta ABD}$,
if P is perimeter of the triangle

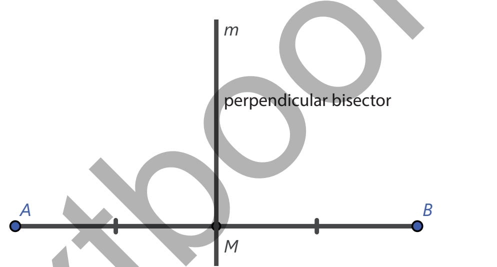
1.4 Median, bisector and altitude of a triangle. Problem solving

1. **Draw following the plan and draw a conclusion.**
   Draw a right triangle. Draw all medians of this triangle.
   a) What can you say about the relative position of the medians?
   b) Do the medians intersect in one point?
   c) Is it true for any triangle? Explain your answer using other triangle.

2. **Draw following the plan and draw a conclusion.**
   Draw an isosceles triangle. Draw its altitudes.
   a) What can you say about relative position of its altitudes?
   b) Do the altitudes intersect in one point?
   c) Is it true for any triangle? Explain your answer using other triangle.

3. **Draw following the plan and draw a conclusion.**
   Draw an obtuse isosceles triangle. Draw its bisectors.
   a) What can you say about relative position of its bisectors?
   b) Do the bisectors intersect in one point?
   c) Is it true for any triangle? Explain your answer using other triangle.

4. **Draw following the plan and draw a conclusion.**
   Draw an acute triangle. Use ruler and set square to draw perpendicular bisector to each side of a triangle.
   a) What can you say about relative position of its perpendicular bisectors?
   b) Do the perpendicular bisectors intersect in one point?
   c) Is it true for any triangle? Explain your answer using other triangle.

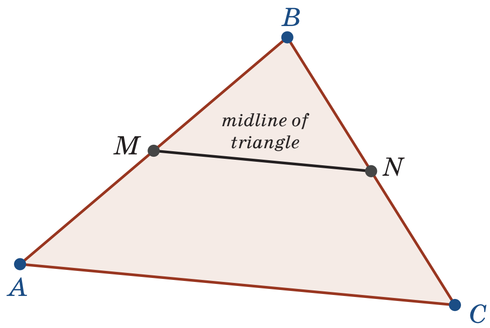


Median, bisector, altitude, and perpendicular bisector are called **remarkable lines of a triangle.** Why?

REMEMBER!

Straight line, perpendicular to some segment, and passing through its midpoint is **called perpendicular bisector.**

5. **Leyla cut a triangle out of thick paper. She says that she can find median, bisector and altitude of the triangle without drawing tools. How can she do this?**



6. **Arman drew a triangle *ABC* and drew midlines of a triangle. How many medlines did he get? Explain your answer.**

REMEMBER!

Midline of triangle is a segment, connecting midpoints of two its sides.

7. **Draw using ruler and protractor.**
Draw a isosceles triangle *ABC* (*AB*=*AC*) with an angle 70° and side *AB* of 6cm. Draw all midlines and measure their lenght. What figure did they form? Find perimeter of the resulting figure.

8. **Cut an equilateral triangle from thick paper. Draw medians of this triangle and find point of their intersection. Put a pencil in this point. What can you notice? Find information about another name of point where medians of triangle intersect.**

9. **AM is median of a triangle *ABC* with a perimeter 20 cm. *AM* divides the triangle into two triangles. Perimeter of triangle *ABM* is 13, and $P_{\Delta AMC}$ = 12 cm. What is length of median *AM*?**

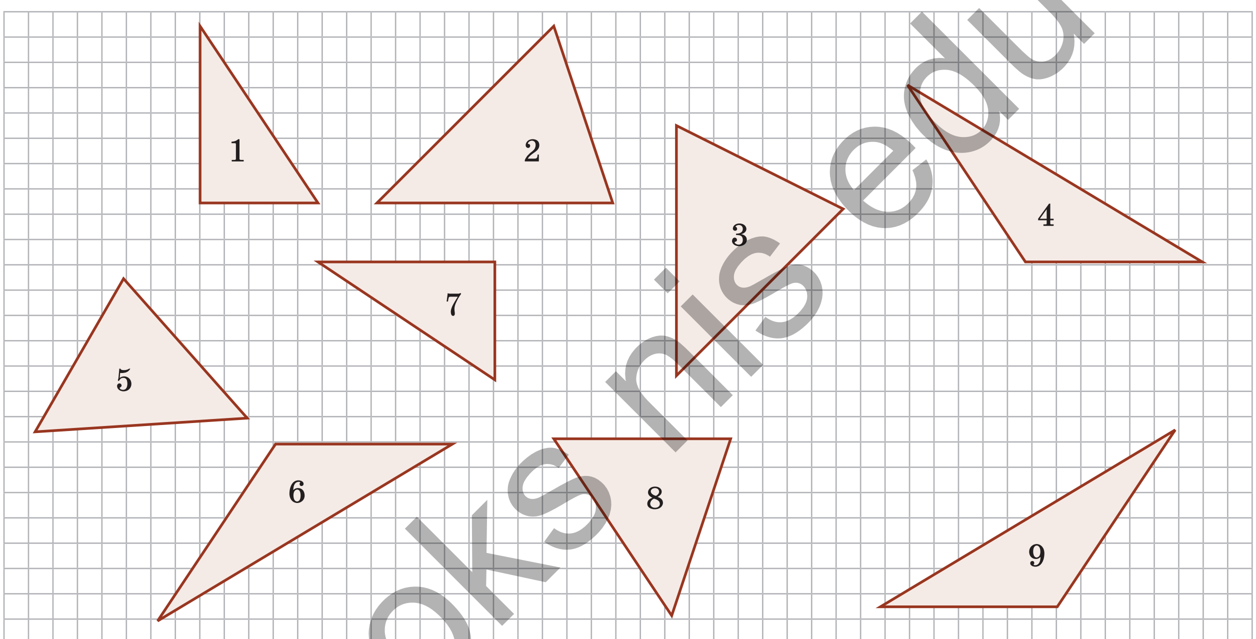
1.5 The first condition of congruence of triangles

You alreay know the concept of congruent figures. Now we going to apply this concept to triangles.

REMEMBER!

Triangles are congruent, if they coincide while putting one on the top of the other. Equal sides and angles are called corresponding.

1. Find congruent triangles among given below. How can you do this? Explain your answer.



Two triangles are congruent if the following conditions are true:

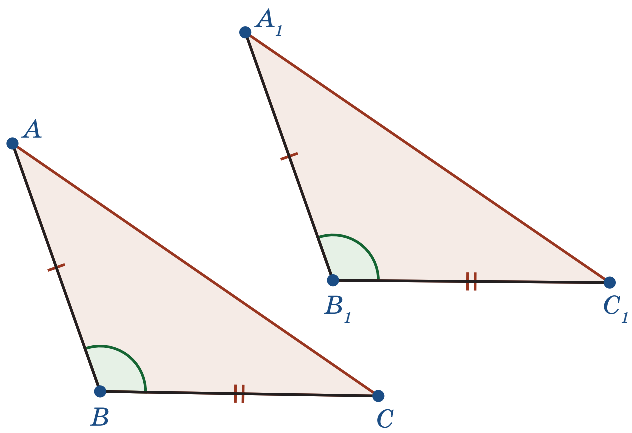
- three sides of one triangle equal to three corresponding sides of another triangle;
- three angles of one triangle are equal to three corresponding angles of another triangle.

Conditions of congruence will help to reduce the number of conditions stated above.

REMEMBER!

If two sides and the adjacent angle of one triangle are respectively equal to two sides and the adjacent angle of another triangle, then these tiangles are congruent.

2. **Read and comment the proof of the first condition of congruence of triangles.**



Given:
$\Delta ABC, \Delta A_1B_1C_1$
$AB = A_1B_1,$
$BC = B_1C_1,$
$\angle ABC, = \angle A_1B_1C_1$

Prove:$\Delta ABC, = \Delta A_1B_1C_1$

Proof:

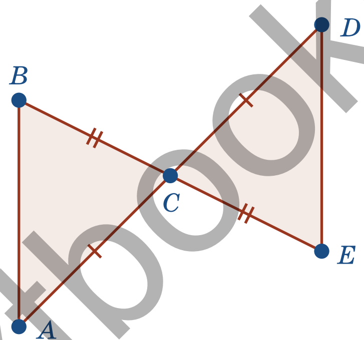
Put ΔABC on top of $\Delta A_1B_1C_1$.

Side BA will coincide with the side B_1A_1, and the side BC — with the side B_1C_1. Given $AB = A_1B_1$ and $BC = B_1C_1$, the point A coincides with the point A_1, and point C — with the point C_1. That means the verices of the triangle will coincide and the triangles ABC and A_1B_1C will be congruent.

Which proves the theorem.

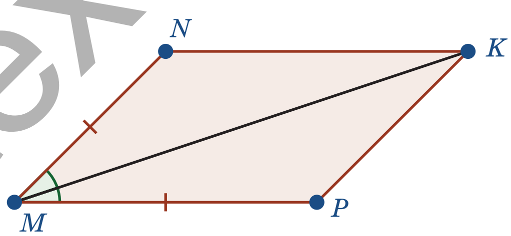
This condition is also called "side-angle-side".

3. **Use the above proved theorem to solve the problems below.**



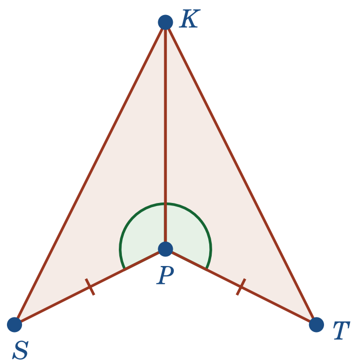
Given:
$AC = CD,$
$BC = CE.$

Prove: $\Delta ABC = \Delta CDE$



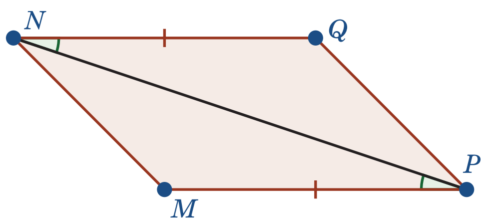
Given:
$MN = MP,$
$\angle MNK = \angle KMP.$

Prove: $\Delta MNK = \Delta KMP$



Given:
$SP = PT$,
$\angle SPK = \angle TPK$.

Prove: $\Delta SPK = \Delta TPK$.



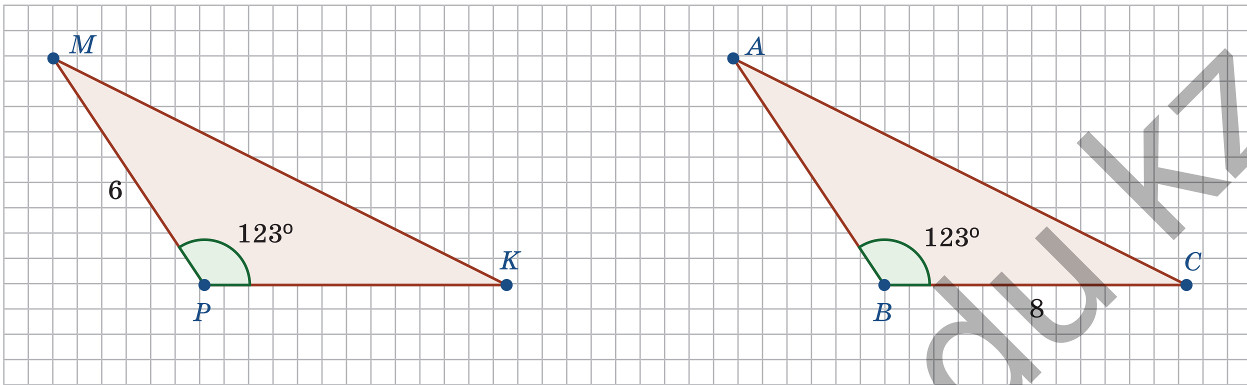
Given:
$NQ = MP$,
$\angle QNP = \angle NPM$

Prove: $\Delta NQP = \Delta NMP$

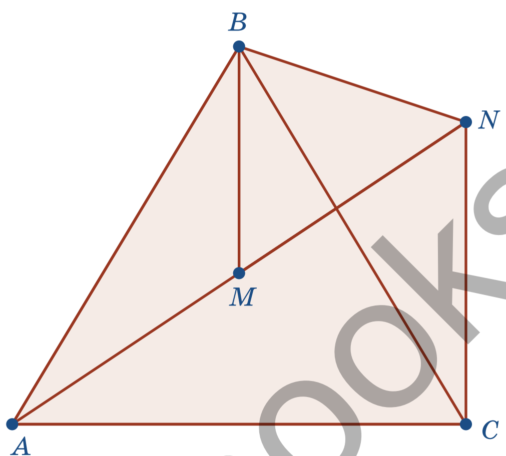
4. **Draw following the plan.**
   a) Draw two segments $AB = 6$ cm, $CD= 7$ so that they intersect at their common midpoint O.
   b) Find the length of the segment BD, if AC is 5 cm.
   c) Write all pairs of corresponding elements of the triangles. Explain your answer.

1.6 The first condition of congruence of triangles. Problem solving

1. **Tomiris drew two triangles and stated that they are congruent. What condition should be added so that her statement is correct?**

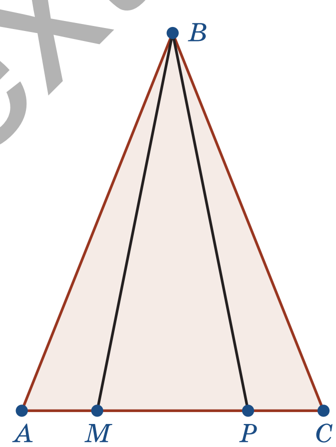


2. **Work with the drowing. Find congruent triangles.**



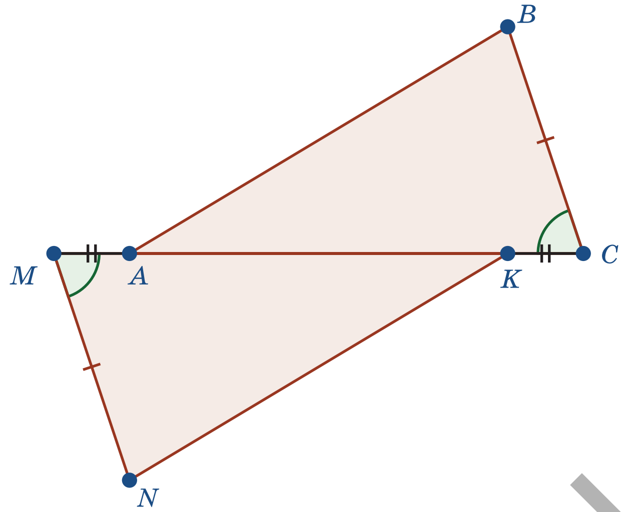
Given:
$AB = BC$,
$BM = BN$,
$\angle ABC = \angle MBN$.

3. **Determine type of triangle *BMP*.**



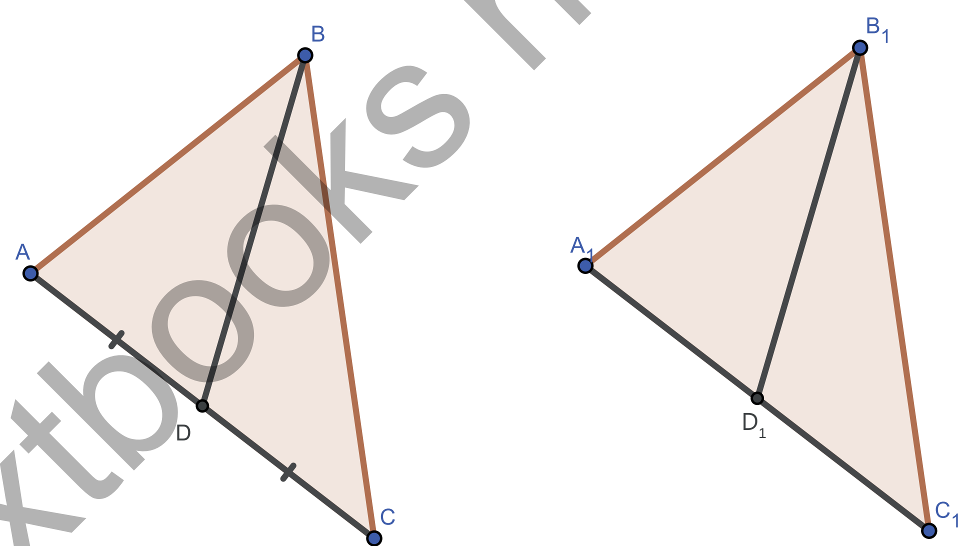
Given:
$AB = BC$,
$AM = PC$,
$\angle BAM = \angle BCP$.

4. Look at the drawing. Is it true that if $AM = KC$, then $AB = KN$? Explain your answer.



5. Sides AB and AC of triangle ABC are equal to 11,6 cm and 18 cm respectively. Median of triangle BM, which length is 7 cm, was extended beyong the point M on segment $MK = BM$. Find perimeter of triangle MKC.

6. Is it true that if triangles are congruent, then medians, drawn to coresponding sides, are also equal? Explain your answer.



1.7 Second condition of congruence of triangles

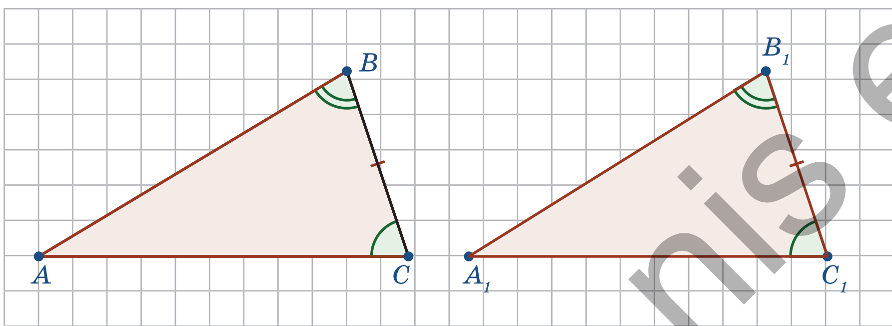
We are going to consider the next condition of congruence of triangles, which is called second condition of congruence or "angle-side-angle".

REMEMBER!

If a side and its two adjacent angles of one triangle are equal to a side and two its adjacent angles of another triangle, then these triangles are congruent.

1. **Comment the proof of second condition of congruence of triangles.**

   To prove the theorem cut out two triangles similar to the given below.



Given:
$\triangle ABC$, $\triangle A_1B_1C_1$,
$BC = B_1C_1$,
$\angle ACB = \angle A_1C_1B_1$,
$\angle ABC = \angle A_1B_1C_1$.

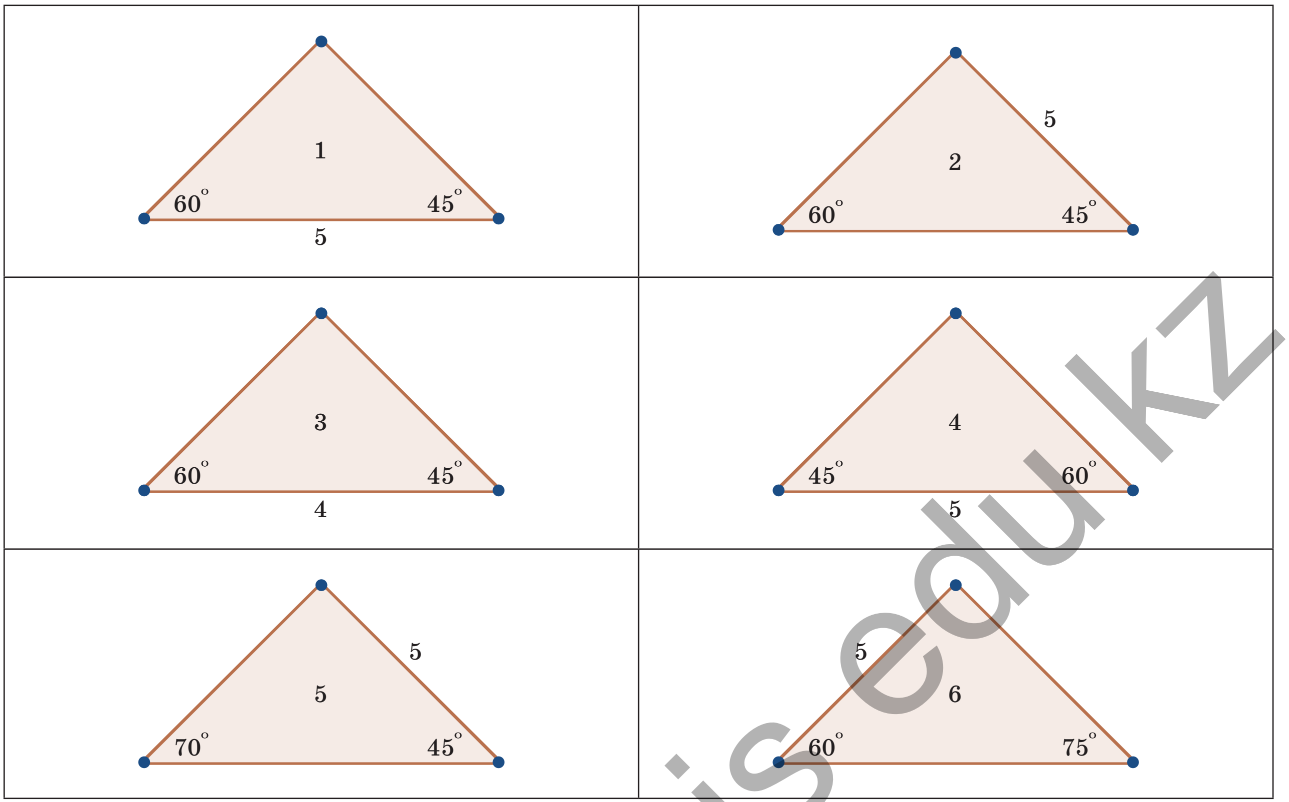
Prove:
$\triangle ABC = \triangle A_1B_1C_1$.

Proof: Put ΔABC on $\Delta A_1B_1C_1$ so that point B coincide with point B_1, and segment BC — with segment B_1C_1 (so that $BC = B_1C_1$). Points A and A_1 should lay on the same side of line BC.

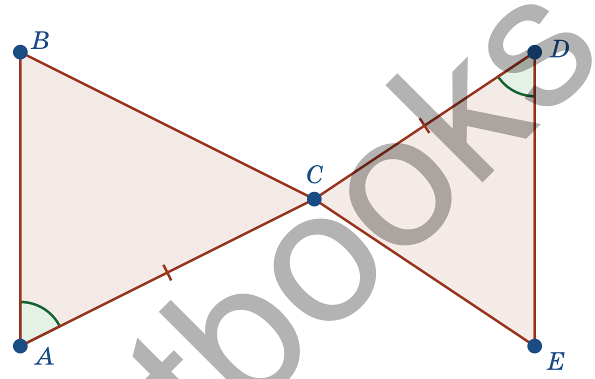
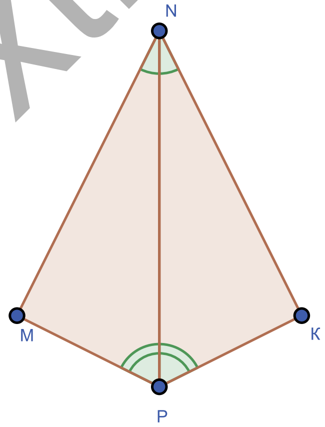
Since $\angle ABC = \angle A_1B_1C_1$ и $\angle ACB = \angle A_1C_1B_1$, then side BA will coincide with side B_1A_1, and side CA — with side C_1A_1. Then point A (the common vertex of sides BA and CA) will coincide with point A_1 (common vertex of sides B_1A_1 and C_1A_1). That means that triangles will fully coincide when one on top of the other, which means they are congruent.

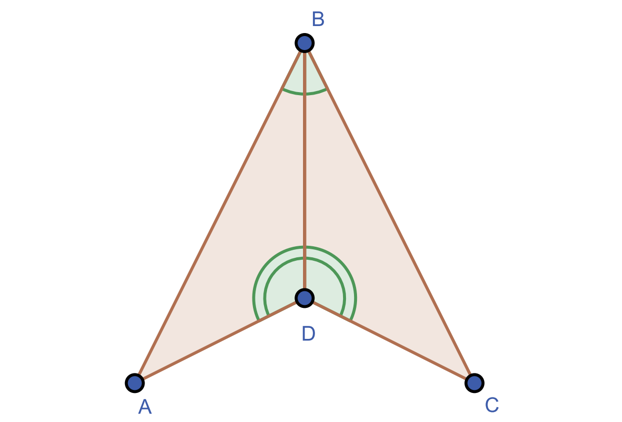
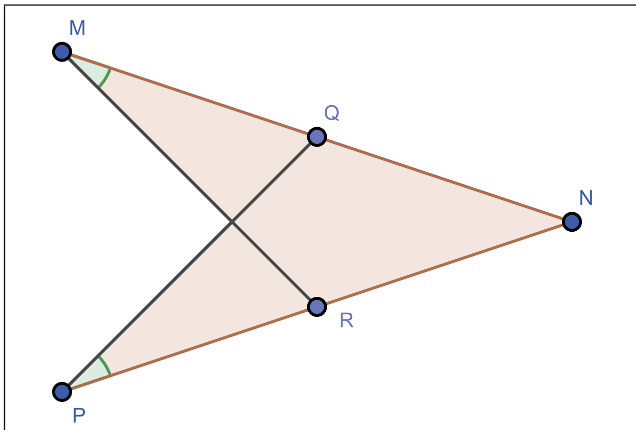
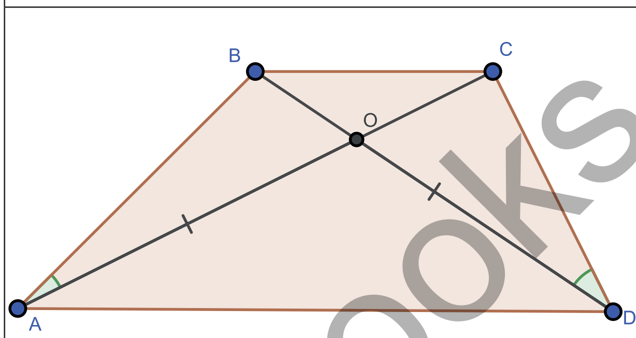
Which proves the theorem.

2. Find congruent triangles, among triangles, given below. Explain your answer.



3. Solve problems, using following drawings.

	**Given:** $AC = CD$, $\angle BAC = \angle CDE$. **Prove:** $\Delta ABC = \Delta CDE$.
	**Given:** $\angle MNP = \angle PNK$, $\angle MPN = \angle NPK$. **Prove:** $\Delta MPN = \Delta NPK$

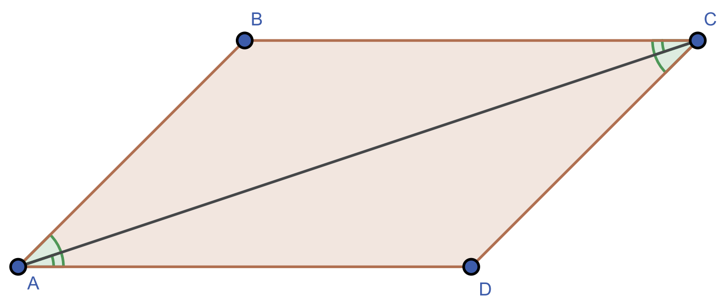
	**Given:** $\angle ABD = \angle DBC$, $\angle ADB = \angle BDC$. **Prove:** $\Delta ABD = \Delta DBC$.
	**Given:** $\angle NMR = QPN$, $MN = PN$. **Prove:** $\Delta MRN = \Delta PQN$
	**Given:** $AO = OD$, $\angle BAC = \angle CDB$. **Prove:** $\Delta AOB = \Delta COD$.

5. **Draw two congruent triangles *ABC* and $A_1B_1C_1$. Draw bisectors of angles *B* and B_1. Are these bisectors equal? Explain your answer.**

1.8 Second condition of congruence of triangles. Problem solving

1. **Asel drew two triangles AOD and COB so that $OD = OC$ as shown in the drawing below. She argues that these triangles are congruent according to second condition of congeuence of triangles. What should she add to her drawing to make the statement correct?**
   What should we add to the drawing to make these triangles congruent according to the first condition of conguence?

2. **Solve the problem.**

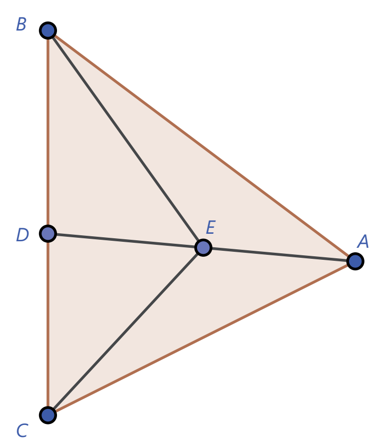


Given:
$\angle BAC = \angle ACD$,
$\angle BCA = \angle CAD$

Prove: $\angle ABC = \angle ADC$.

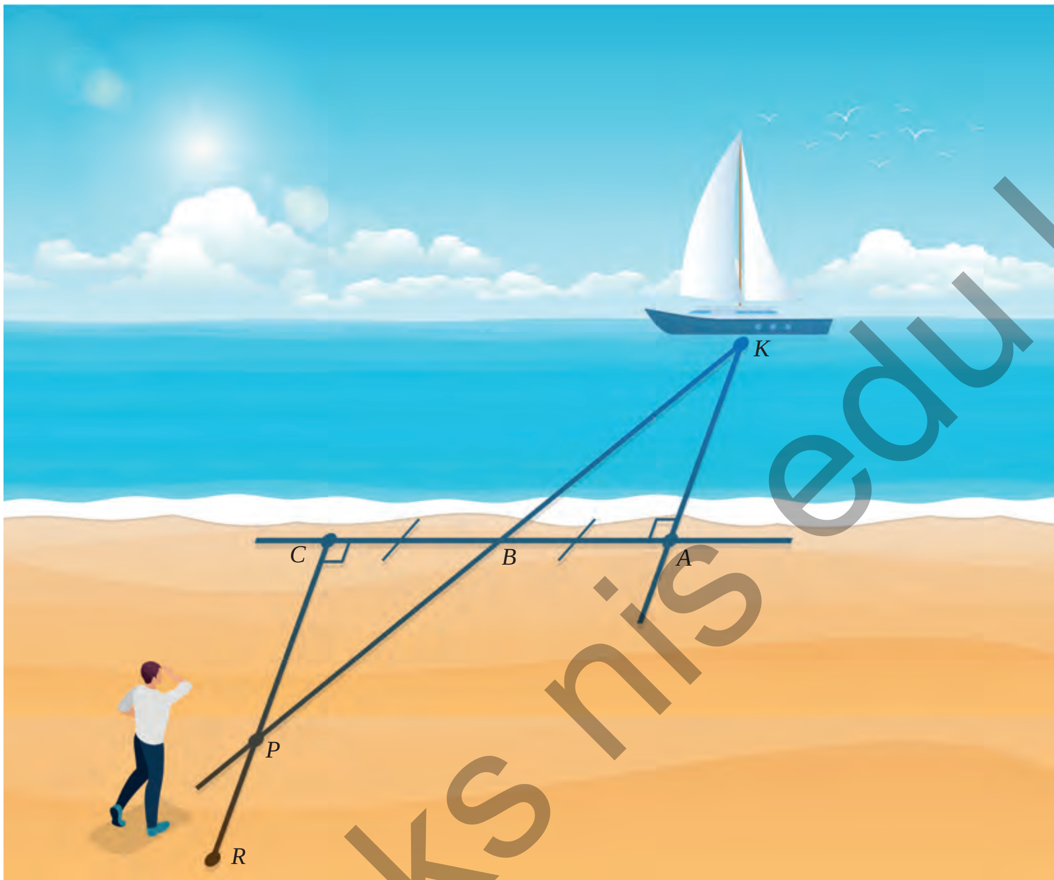
3. **Draw following the plan and answer the questions.**
   a) Draw two parallel lines m and n.
   b) Mark the point M on the line m, the point N on the line n so that the segment MN forms the angles equal to 50^0 with the given lines.
   c) Find the midpoint of the segment MN and mark it A.
   d) Draw a straight line through the point A that intersects the lines m and n at the points P and Q respectively.
   Is point A the midpoint of PQ? Why? Explain your answer.

4. **Leila draw triangles as shown in the drawing below where $\angle BED = \angle DEC$, $\angle BDE = \angle EDC$. What can you say about the triangles depicted in the picture? Explain your answer.**



5. **Range finder was built in the harbor of Miletus, which determined the distance from shore to a ship. It consisted of three pegs placed at the same distance from one another (in the picture below, they are indicated by points A, B and C).**

When a ship appeared on the horizon, observers started lockating a point *P* on a line *CR*, so that points *K*, *B* and *P* were collinear. The length of the segment *CP* was equal to the length from the shore to the ship. Why? Explain your answer.



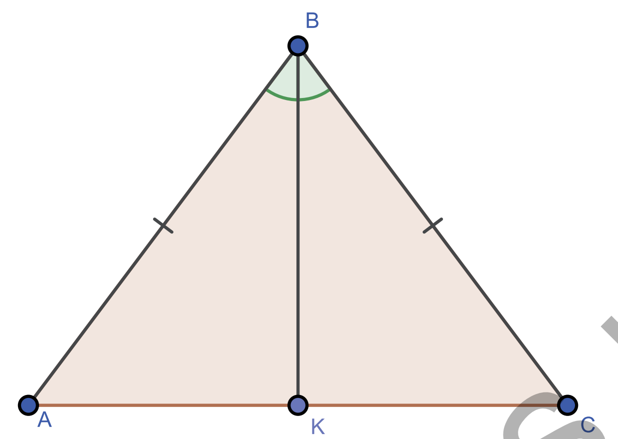
1.9 Isosceles triangle and its properties

You already know that there are all kinds of triangles (acute, obtuse etc.), but we should pay special attention to isosceles triangle, as its properties will be useful for further problem solving.

REMEMBER!

- Base angles are equal in isosceles triangle
- Bisector, drawn to its base, is median and altitude.

1. **Look closer at the proof of properties of isosceles triangle and give your comments.**



Given:
ΔABC
$AB = BC$

BK — bisector

Prove:
1) $\angle A = \angle C$;
2) $BK \perp AC$, $AK = KC$.

Proof:

$AB = BC$ (given),
BK — common side,
$\angle ABK = \angle KBC$ (since BK is bisector of angle ABC)

$\Rightarrow \triangle ABK = \triangle BKC$

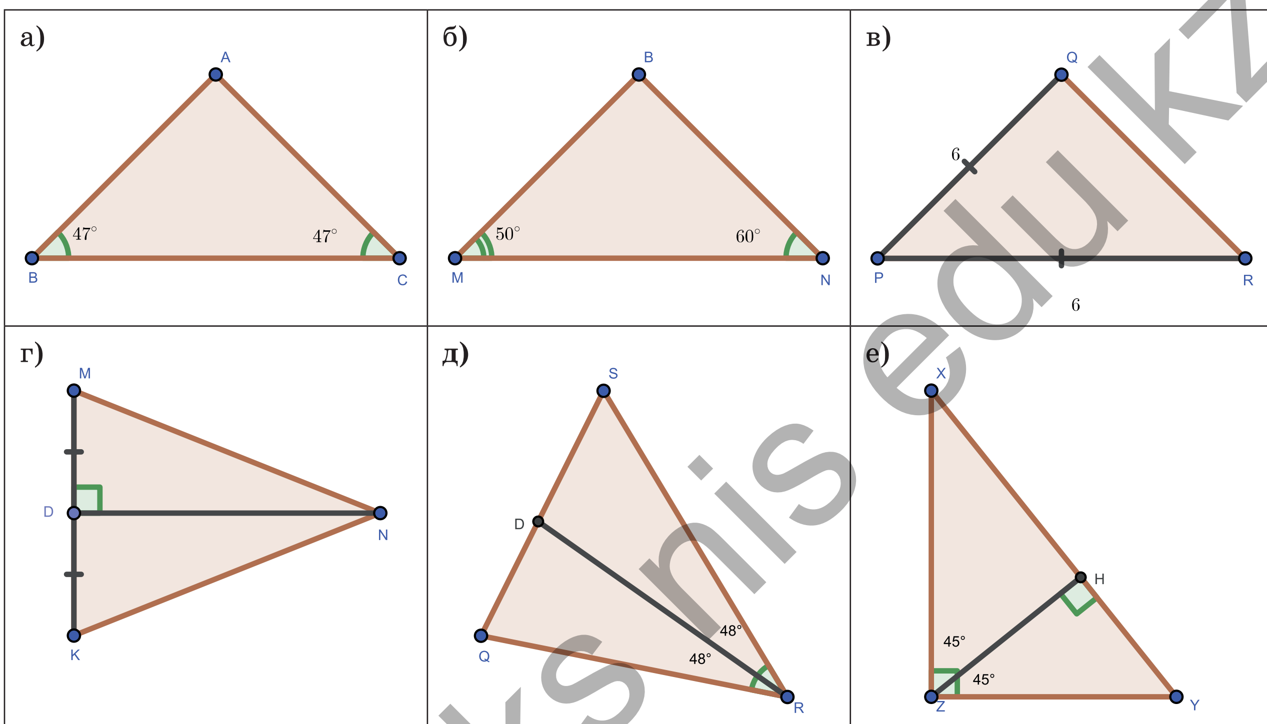
(by the first condition of congruence of triangles).

That means that,
1) $AK = KC$,
2) $\angle A = \angle C$, $\angle BKA = \angle BKC = 90^0$ (since these angles are supplementary adjacent they add up to 180^0), hence, BK is a median and an altitude of the triangle ABC.

Which proves the theorem.

2. **Prove that if:**
   a) bisector and altitude drawn from the same vertex of a triangle coincide, the triangle is isosceles;
   b) altitude and median, drawn from the same point coincide, this triangle is isosceles;
   c) two angles are equal, the triangle is isosceles;

3. **Refer to conditions of isosceles triangle to find isosceles triangles among given below. Explain your answer.**

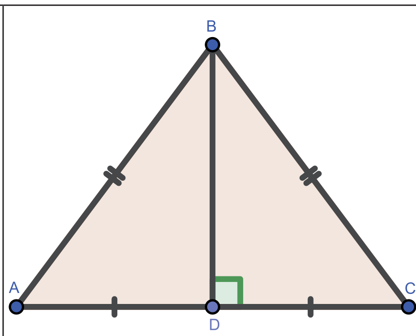


4. **All properties of isosceles triangle are applicable to equilateral triangle. Is this statement correct? Explain why.**

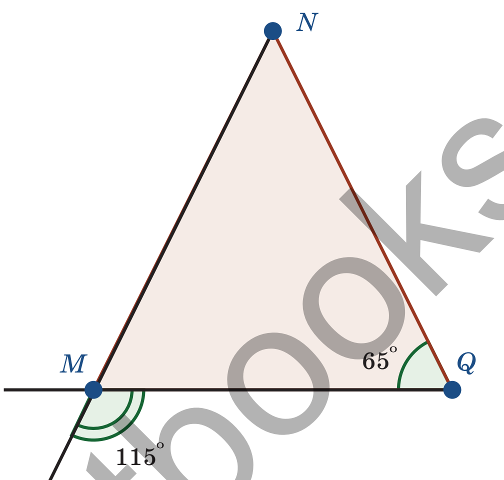
5 **All angles of equilateral triangle are equal. Is this statement correct? Explain why.**

1.10 Problem solving

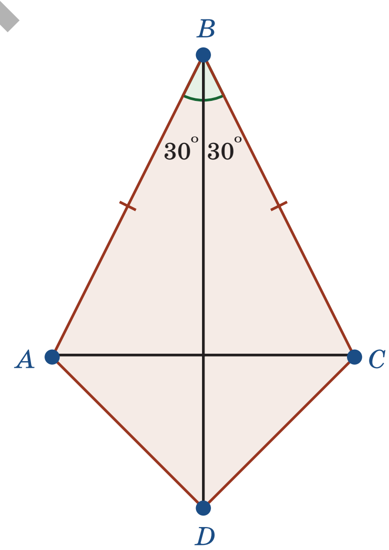
We are going to apply propeties of isosceles triangle in problem solving, but first revise both properties and conditions of isosceles triangle.

Properties of isosceles triangle Isosceles triangle has: • coinciding median, bisector and altitude drawn from the vertex opposite to the base that. • equal angles adjacent to the base.		**Conditions of concruent triangles** If a triangle has: • two equal angles, then the triangle is isosceles; • altitude that coincides with median, **then the triangle is isosceles;** • bisector that coincides with median, **then the triangle is isosceles;**

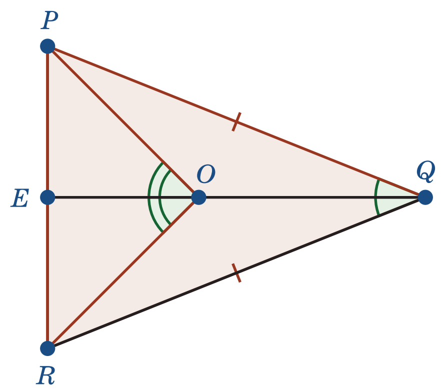
1. **Use the ready made drawings and apply properties of isosceles triangle to solve the problems.**



Given: ΔMNQ
Prove: ΔMNQ is isosceles.

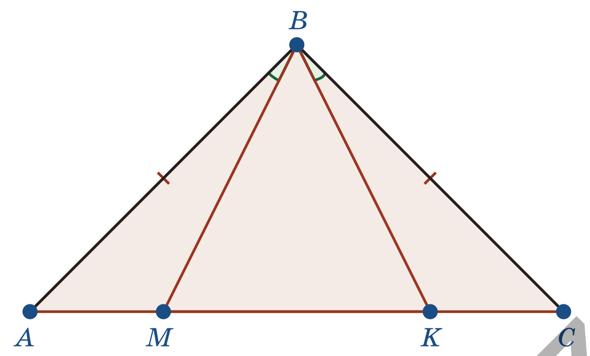


Given:
$AB = BC$
$\angle ABD = \angle BCD = 30°$
Prove: $\triangle ADC$ isosceles.



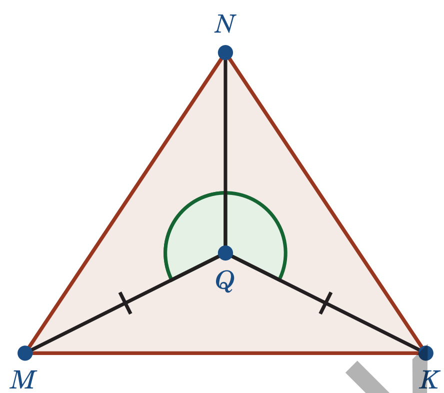
Given:
$PQ = QR$
$\angle PQO = \angle OQR$

Prove: $\Delta\, QPR$ isosceles



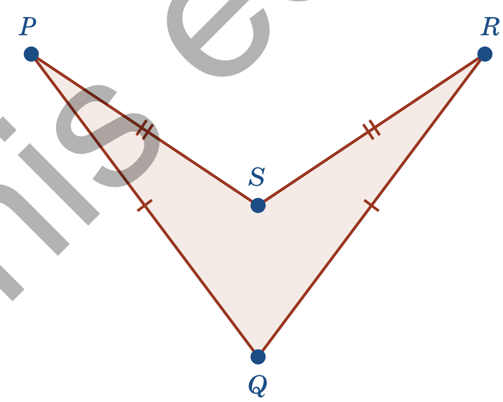
Given:
$AB = BC,$
$\angle ABM = KBC$

Prove: $\Delta\, MBK$ isosceles



Given:
$MQ = QK$
$\angle NQM = NQK$

Prove: $\Delta\, NMK$ isosceles



Given:
$PQ = QR$
$PS = RS$

Prove: $\angle QRS = \angle QPS$

2. Is it true that in an isosceles triangle medians drawn to the lateral sides are equal? Explain your why.

3. Draw following the plan and answer the questions.

a) Draw an isosceles triangle ABC, where $AB = BC$.
b) On the outer sides of AB and BC draw equilateral triangles ABK and BCM.
c) Connect the vertices of equilateral triangles (other than the vertices of an isosceles triangle) with the point N – the midpoint of the AC.
d) Determine the type of triangle MKN. Explain your solution.

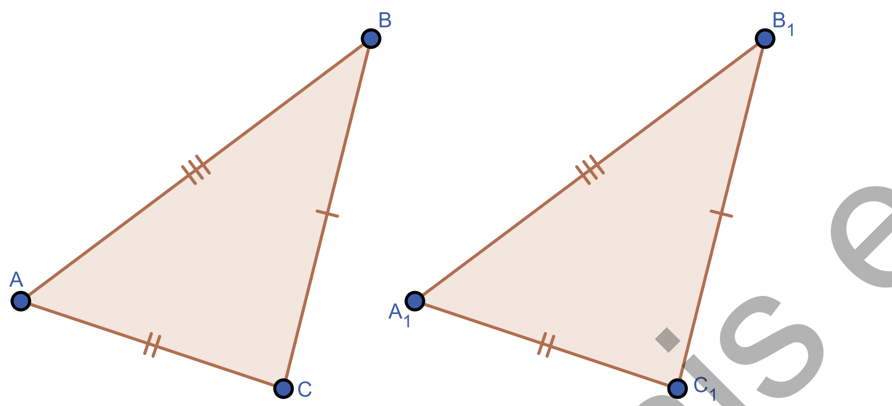
1.11 Third condition of congruence of triangles

Consider another condition of congruent triangles, which is called the third condition of congruent triangles, sometimes it is also called "side-side-side."

REMEMBER!

If three sides of one triangle are respectively equal to three sides of another triangle, then these triangles are congruent.

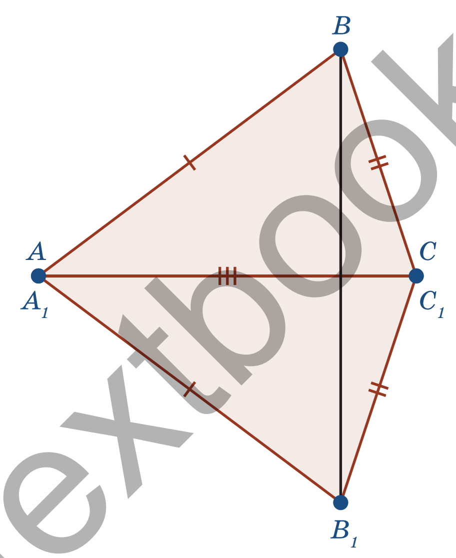
1. **Comment on the proof of the third condition of congruence of triangles.**



Given:
$AB = A_1B_1$
$BC = B_1C_1$
$AC = A_1C_1$

Prove:
$\Delta ABC = \Delta A_1B_1C_1$

Proof:



1. Place triangles ABC and $A_1B_1C_1$ so that one of their sides coincide and points B and B_1 are on opposite sides of the coinciding lines.

2. Triangles ABB_1 and BCB_1 isosceles (Why?), hence,

$\angle ABB_1 = \angle AB_1B$, $\angle B_1BC = \angle BB_1C$,

$\left.\begin{aligned}\angle ABC = \angle ABB_1 + \angle B_1BC,\\ \angle AB_1C = \angle AB_1B + \angle BB_1C.\end{aligned}\right\} \Rightarrow$

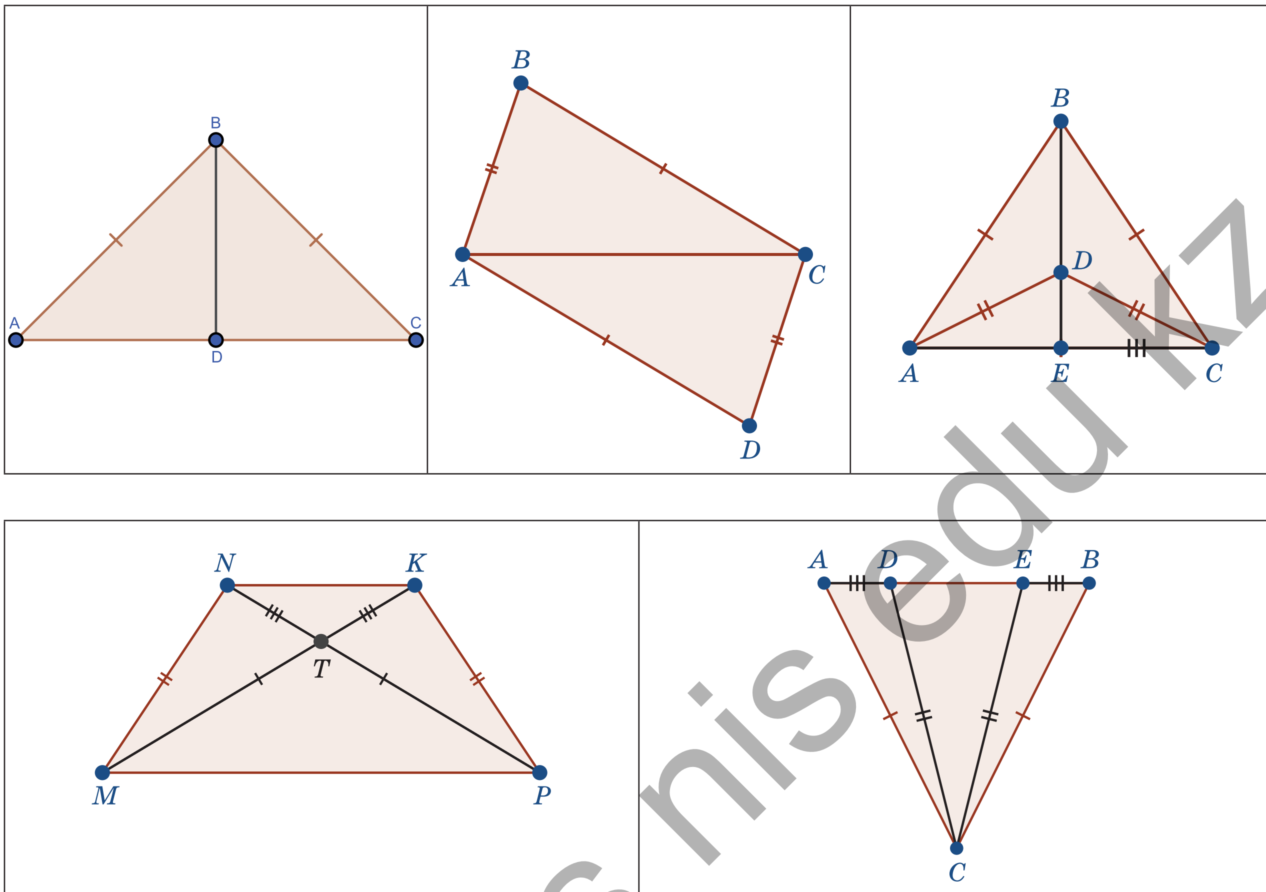
$\Rightarrow \angle ABC = \angle AB_1C.$

$\left.\begin{aligned}AB = A_1B_1 \text{ (by drawing)},\\ BC = B_1C_1 \text{ (by drawing)},\\ \angle ABC = \angle AB_1C\end{aligned}\right\} \Rightarrow$

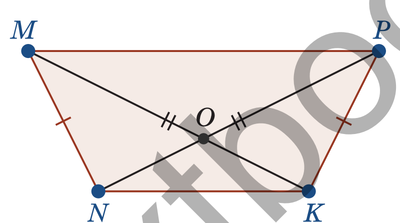
$\Rightarrow \Delta ABC = \Delta A_1B_1C_1$ (Why?)

Which proves the theorem.

2. **Apply the third condition of congruence of triangles to find congruent triangles among given below.**

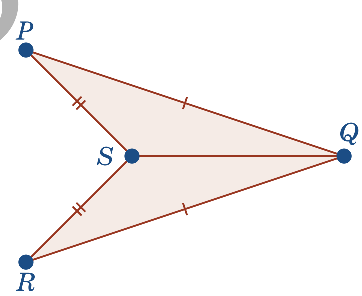


3. **Solve problems using ready made drawings.**



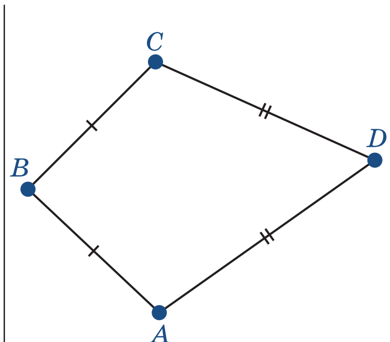
Given:
$MN = KP$,
$MO = OP$.

Prove: $\angle NMP = \angle KPM$



Given:
$PQ = QR$,
$PS = RS$.

Prove: $\angle QPS = \angle QRS$



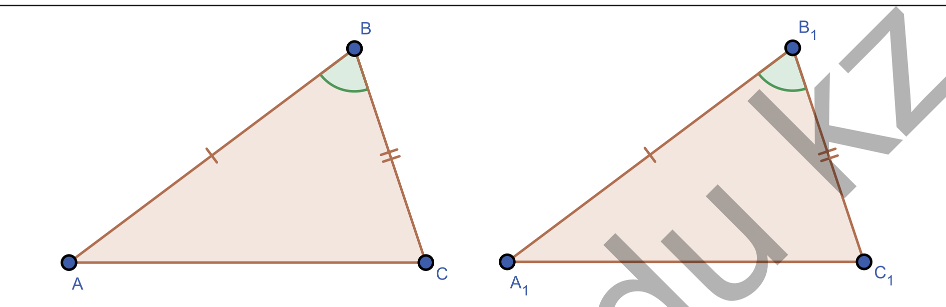
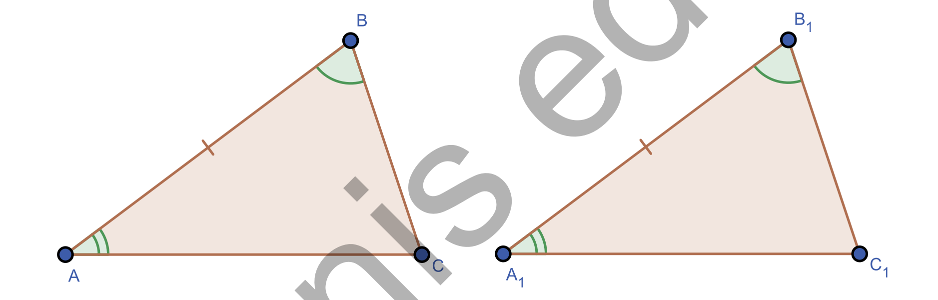
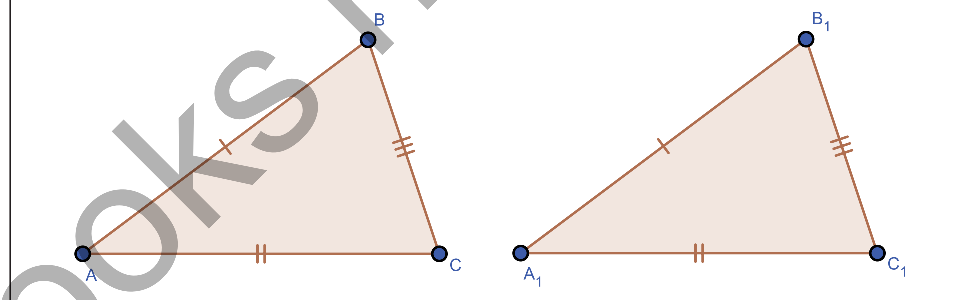
Given:
$AB = BC$,
$AD = CD$.

Prove: $\angle A = \angle C$.

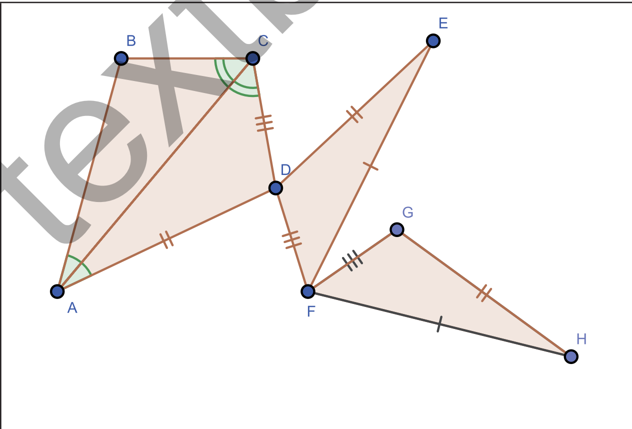
1.12 Conditions of congruence of triangles. Problem solving

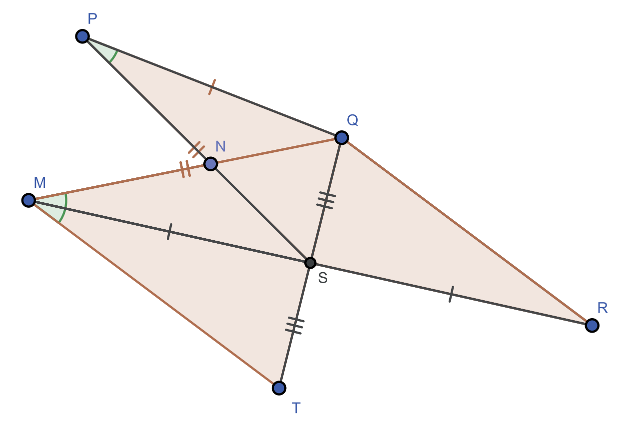
Now you know how to compare triangles, and it's time to apply your knowledge.
First revise the conditions of congruence of triangles you know.

1. Formulate of conditions of congruence of triangles.

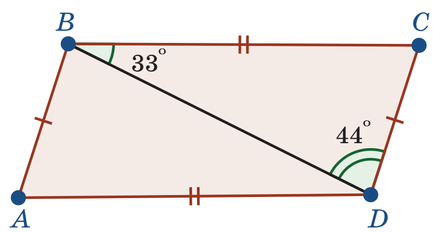
The first condition of congruence of triangles "Side-angle-side"	
The second condition of congruence of triangles "Angle-side-angle"	
The third condition of congruence of triangles "Side-angle-side"	

2. Work with the drawing. Find congruent triangles. Explain your answer.

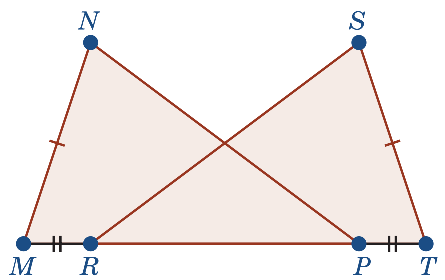




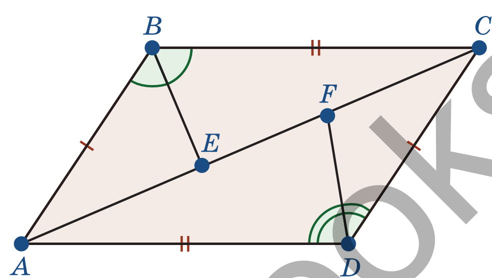
2. **Given triangles ABC and MNK with AB=14 cm, BC=10 cm, NK=10 cm. What conditions should be added to make these triangles congruent. Explain why.**
   a) $MN = 14$ cm, $\angle ABC = \angle MNK = 50^{\circ}$;
   b) $MN = 14$ cm, $\angle BAC = \angle NMK = 50^{\circ}$;
   c) $MN = 14$ cm, $\angle ABC = \angle MKN = 50^{\circ}$;
   d) $MN = 14$ cm, $AC = MK = 15$ cm;
   e) $MN = 14$ cm, $\angle NMK = \angle BAC = m^{\circ}$, $\angle MNK = \angle ABC = n^{\circ}$.
3. **Use ready made drawings to find the unknown elements of a triangle.**



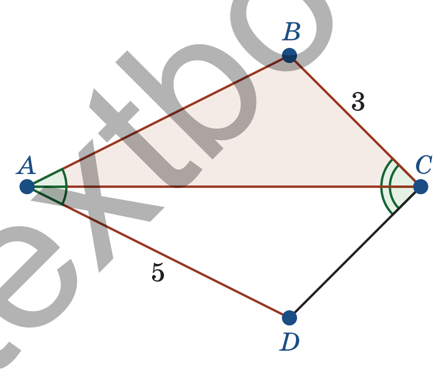
Given:
$AB = CD$,
$BC = AD$,
$\angle CBD = 33°$,
$\angle BDC = 44°$.
Find: $\angle ABD$.



Given:
$MN = ST$,
$MR = PT$,
$NP = SR$.
$\angle NMT = 45°$.
Find: $\angle STP$.



Given:
$AB = CD$,
$BC = AD$.
BE — angle bisector of ABC,
DF — angle bisector of ADC.
$BE = 17$
Find: DF.



Given:
$\angle BAC = \angle CAD$,
$\angle BCA = \angle ACD$,
$BC = 3$ cm,
$AD = 5$ cm.
Find: P_{ABCD}.

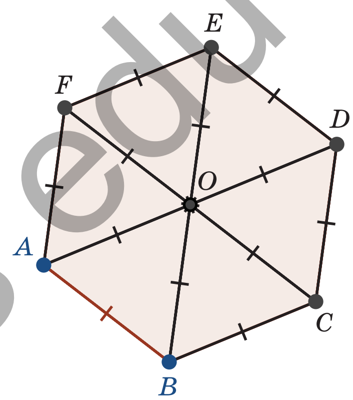
4. **Damir drew two congruent triangles MNP and MQR so that $MP = MQ = 5$ см, $MN = 6$ cm, and $MR = 10$ cm. What are the lengths of RQ and NP?**
5. **Draw a figure so that it can be divided into:**
   a) into two congruent triangles;
   b) into three congruent triangles.

How can you make it? Explain why.

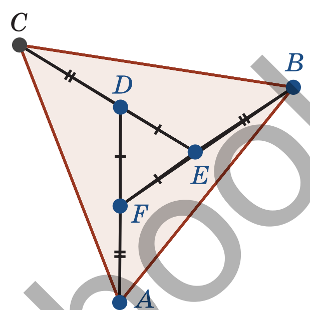
1.13 Problem solving

1. **What statements are correct among given below? Why? Draw to illustrate your answer.**
   a) if $AB = MN$, $AC = MK$ and $\angle BAC = \angle NMK$, then $\Delta ABC = \Delta MNK$
   b) if $AB = MN$, $\angle BAC = \angle NMK$, $\angle CAB = \angle MNK$, then $\Delta ABC = \Delta MNK$
   c) if $AB = MN$, $BC = NK$, $AC = MK$, then $\Delta ABC = \Delta MNK$
   d) if $\Delta ABC = \Delta MNK$, then $AB = MN$

2. **Given triangle *ABC*. Points *M* and *N* were marked on the sides *AC* and *BC* respectively. Determine the type of the triangle *ABC* if $\Delta ANB = \Delta AMB$. Given that the corresponding sides of the triangles are:**
   ***AB* and *AM*, *BN* and *AB*, *AN* and *BM*.**
3. **Is it true that the midpoints of the sides of an isosceles triangle are the vertices of another isosceles triangle? Why? Explain your answer.**
4. **Aliya has six equilateral congruent triangles. she can arrange them into a figure like given in the drawing. How many triangles congruent to *AED* she can find? Why are they congruent? Explain your answer.**



5. **Solve the problem using the ready made drawing.**



Given:
ΔDEF **— equilateral**

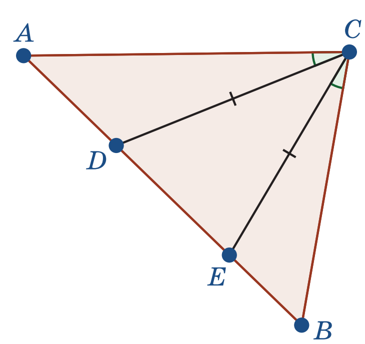
$AF = CD = BE$.

Prove: Δ ***ABC*** **equilateral.**

6. **Given triangle *ABC* with $CD = CE$, $\angle ACD = \angle ECB$, and a perimeter 42,9 cm. Determine the type of the triangle *ABC* and find the length of its sides if one of the sides is $1\frac{2}{3}$ the other.**

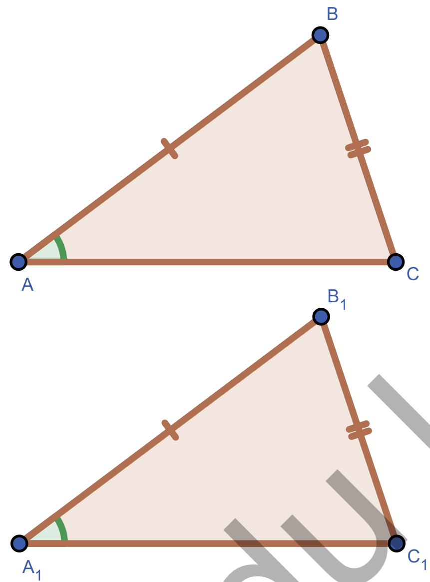
   Find one more pair of congruent triangles. Why are they congruent?

How many ways can you find prove it?



7. **Prove that if the two sides and the angle opposite to the larger side of one triangle are respectively equal to the two sides and the angle opposite to the larger side of the other triangle, then these triangles are congruent.**

The definition of this problem is sometimes called the **fourth condition of congruence of triangles.** When proving it, you can also use the definition of congruent figures and compare these triangles by fitting one on the top of the other or by supremeposing.



8. Mark off three half lines OM, ON and OK from the point O, so that the half line ON is the bisector of the angle MOK and the segments MN and NK are equal. Is it true that the triangles MON and LCM are equal?

1.14 What have I learned?

Fill gaps in the table in order to assess your knowledge about the unit.

TRIANGLE

Triangle
- Triangle is ...
- Isosceles triangle is ...
- Equilateral triangle is ...
- Acute triangle is ...
- Obtuse triangle is ...

Remarkable lines of triangle
- Median of triangle is ...
- Altitude of triangle is ...
- Bisector of triangle is...
- Perpendicular bisector of triangle is ...
- Median line of triangle is ...
- Medians of triangle intersect ...
- Altitudes of triangle intersect ...
- Bisectors of triangle intersect ...
- Perpendicular bisectors of a triangle intersect ...

Conditions of congruence of triangles
- If two sides and angle between them ...
- If side and its two adjacent angles ...
- If three sides of one triangle ...

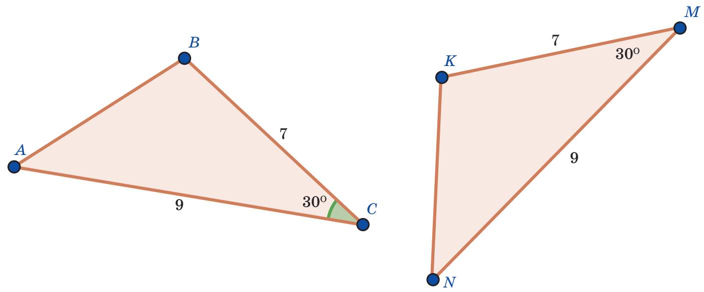
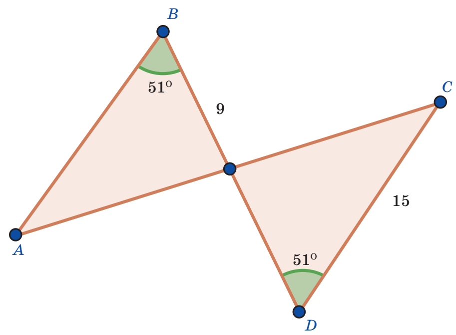
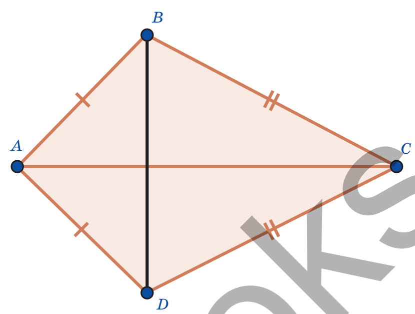
Isosceles triangle
- Two sides are ... in isosceles triangle
- Bisector and median are ... in isosceles triangle
- If angles in a triangle...
- If median in a triangle is ...
- If bisector in a triangle is ...

Questions that help you revise the material.
Make sentences, using following words at least ones:
- triangle;
- equilateral triangle;
- isosceles triangle;
- acute triangle;
- median of triangle;
- bisector of triangle;
- altitude of triangle;
- conditions of congruence of triangles;

1. **Are the following statements correct? Explain why.**
   a) If two triangles are congruent and one of them is isosceles, then the other is also isosceles.
   b) If two triangles are congruent and one of them is acute, then the other is right.

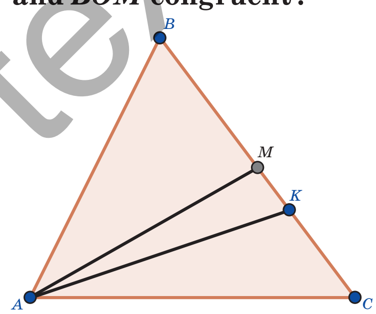
c) If two triangles are congruent and one of them is equilateral, then the other is obtuse.
d) If two triangles are congruent and one of them is obtuse, then the other is isosceles.

2. Solve the problems using ready made drawings.

	**Given:** $\angle BC = MK = 7$, $AC = MN = 9$, $\angle ACB = \angle NMK = 30^{o}$. **Prove:** $\Delta ABC = \Delta MNK$.
	**Given:** $\angle B = \angle D = 51^{o}$, $BD = 18$, $OB = 9$, $DC = 15$. **Find:** AB.
	**Given:** $AB = AC$, $BD = CD$. **Prove:** $BD \perp AC$

3. **The perimeter of an isosceles triangle is 22.4 cm, and its two sides have aspect ratio 2: 3. Find the sides of the triangle.**

4. **Given the equilateral triangle *ABC* where the medians *AM* and *BK* intersect at the point *O*. Are the triangles *AOK* and *BOM* congruent?**



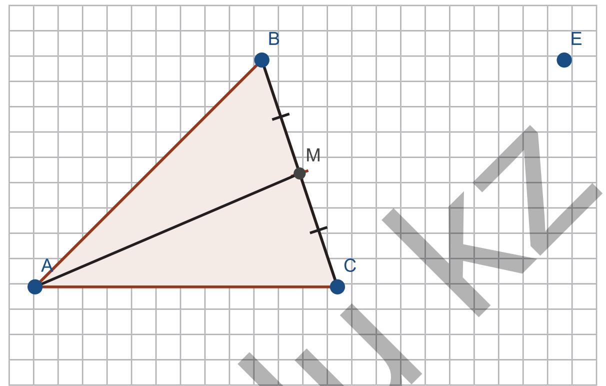
5. **Given the triangle *ABC* with the median *AM*. Point *K* is marked on the segment *CM*, so that *KB*: *KM* = 4: 1. What is the ratio of *KM*: *MB*? *MK*: *KC*? *MK*: *BC*?**

6. **Given the triangle *MNK*, the median drawn from the vertex *N* is equal to the side *NK*. The median divides *MK* into two parts. What is the ratio of that division?**

7. **Given the triangle *MNK*, where the height *NH* divides the angle *MNK* in half. What is the length of median *MB* if *KA* is 15?**

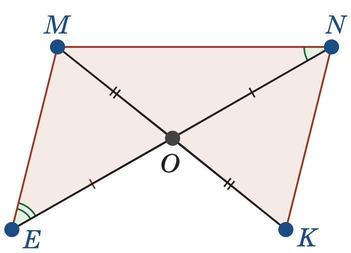
1.15 What have I learned? Self-assessment activities

1. Given two equal triangles ABC and MNK where $AB = 13$ cm, $BC = 12$ cm, $MK = 21$ cm. What is the perimeter of the triangle MNK?
2. Arman extended the median AM of the triangle ABC beyond the point M by the segment $AM = ME$. Than he connected the point E with the vertices B and C of the triangle ABC. How many pairs of equal triangles did he make? Why are they congruent? Explain your answer.



3. Solve the problems:

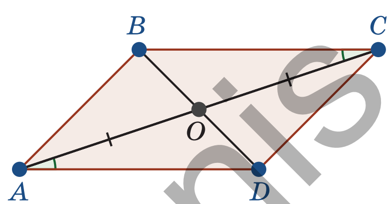
a)



Given:
$OE = ON$,
$OM = OK$,
$\angle MEN = 55°$
$\angle MNE = 37°$.

Find: $\angle MNK$.

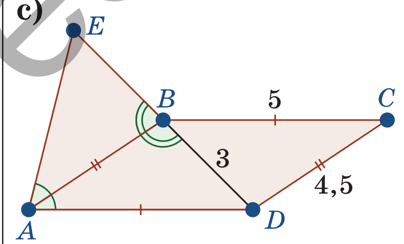
b)



Given:
$OA = OC$,
$\angle CAD = \angle BCA$,
$AC = 16$ cm,
$BC = 14$ cm,
$OB = 0{,}5\,AC$.

Find: $P_{\Delta BOC}$.

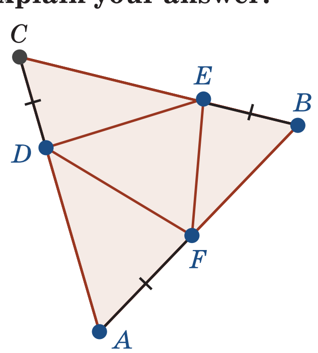
c)



Given:
$AD = BC = 5$,
$AB = CD = 4{,}5$,
$BD = 3$,
$\angle BAE = \angle DAB$,
$\angle ABE = \angle ABD$.

Find: $P_{\Delta AED}$.

4. Given the isosceles triangle ABC ($AB = BC$) with the altitude BH and the point O marked on BH. Is it true that the triangles AOB and COB are congruent? Why? Explain your answer.



5. Given an equilateral triangle ABC. Points F, E, and D are marked on the sides AB, BC and AC respectively, so that $CD = AF = BE$. Determine the type of the triangle DEF.

2 Formulas of abridged multiplication

By the end of this unit, you will have learned:

- ✔ the formulas of abridged multiplication and what they are for;

I will be able to:

- ✔ factor algebraic expressions by taking out the common factor;

factor polynomials by grouping;

- ✔ use the formulas of abridged multiplication when simplifying algebraic expressions;
- ✔ use the formulas of abridged multiplication when factoring.

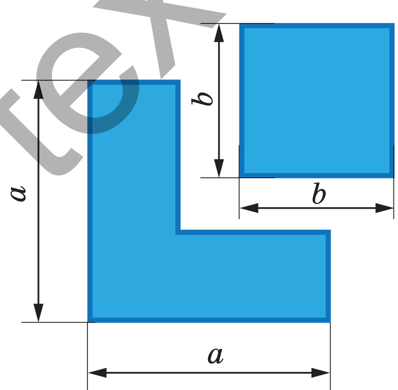
Blaise Pascal

19.06.1623 — 19.08.1662
Place of birth: Clermont-Ferrand, Auvergne
Scientific field: **Mathematics,** Mechanics, Philosophy, Literature, Physics

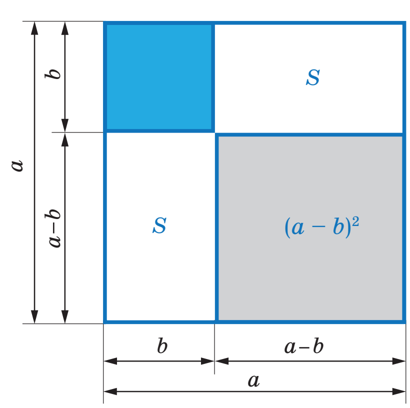
					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+6ab^4+b^5$$

$a^2 - b^2 = (a - b)(a + b)$ $\quad (a + b)^2 = a^2 + 2ab + b^2$ $\quad (a - b)^2 = a^2 - 2ab + b^2$



$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$



2.1 Factoring. Taking out the common factor

Previously, you have learned the concepts of monomials and polynomials, and how to add, subtract and multiply them. Now it is time to master another operation - factoring. This will help you to solve many algebraic problems.

The simplest factorisation method is taking out the common factor.

REMEMBER!

Factoring means presenting a polynomial as a product of polynomials.

If each term of a polynomial has a common factor, it can be put outside the brackets according to distributive law of multiplication.

This can be presented schematically as follows:

$$\Delta \square - \Delta \lozenge = \Delta (\square - \lozenge)$$

1. Take out the common factor:

a) $4a^2 - 8a^4 + 12 = 4(\square - 2\square + \square)$;
b) $4a^2 - 8a^4 + 15a = \square(\square - \square + 15)$;
c) $3x + 6xy + 15x^2 = \square(\square + 2y + \square)$;

d) $a^{3n} - 2a^n + 3a^{2n+1}$, **where** $n \in N$.

2. Arman believes that it is easier to calculate the examples given below by taking out the common factor. Is it true? How can he do that?

a) $126^2 + 126 \cdot 74$; **b)** $145^2 - 145 \cdot 45$;
c) $0{,}4^3 + 0{,}4^2 \cdot 0{,}6$; **d)** $0{,}8^3 - 0{,}64 \cdot 3{,}8$.

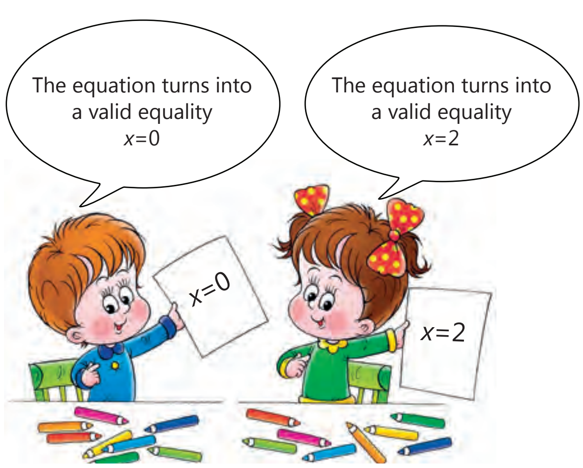
3. Take out the common factor:

a) $a(x-y) + b(x-y)$; **b)** $c(a-b) + 2(a-b)$;
c) $2n(p+c) + 4(c+p)$; **d)** $b(3-x) + a(3-x)$.

Example: take out the common factor:
$5ab - 35bc = 5ab - 5 \cdot 7bc = 5b(a - 7c)$
$x(a-3) + y(a-3) = (a-3)(x+y)$

4. Take out the common factor:
a) $2(a-b) + x(b-a)$;
b) $m(a-b) + x(b-a)$;
c) $2n(p-c) + k(c-p)$;
d) $5c(n-m) - a(m-n)$;
e) $3a(x-y) - 6a^2(y-x)$.

5. Dima and Madina solved the following equation $x(x-2) = 0$. Dima claims that the root of this equation is 0, and Madina believes that this number is 2. Who is right? Why? Explain your answer.



Methods of factorisation, in particular taking out the common factor is suitable for solving equations.

The expression ab equals 0, only if one of the factors equals 0, i.e. $a = 0$ or $b = 0$.

6. Solve the equations using the above mentioned algorithm:

a) $x(2x - 5) = 0$; b) $(x + 4)(x - 7) = 0$;
c) $m(2m - 7)(3m + 4) = 0$; d) $p^3(4 - 2p)(5 + p) = 0$;
e) $(4y - 3)(8y + 2)(5 + 15y) = 0$; f) $(x + 2)(2 - x)(6 - 3x) = 0$.

7. Arman was solving equations on a blackboard, but one part of his solution was erased. Help Arman to restore his solution:

$3x^2 + 6x = 0;$
$3x(\square + 2) = 0;$

$3x = 0;$ *or* $\square + 2 = 0;$
$x_1 = \square$ $\quad \square = \square.$

Answer: $x_1 = \square;\ x_2 = \square.$

8. Solve the equations:

a) $x^2 - x = 0$; b) $3x^2 - 9x = 0$;
c) $2x + 8x^2 = 0$; d) $x^2 = 2x$.

9. Represent a polynomial $p(x)$ as a product of a polynomial and monomial. Find the values of x at which the equation $p(x) = 0$ is valid, if:

a) $p(x) = 6x^2 + 12x$; b) $p(x) = x^2 - 3x^3$;

10. It is known that at some value of a-variable, the value of expression $a^2 - 3a + 2$ equals 9. What will be the value of the following expressions with the same value of a-variable:

a) $2a^2 - 6a + 4$; **b)** $a^2(a^2 - 3a + 2) - 3a(a^2 - 3a + 2)$; **c)** $4a^2 - 12a - 10$?

11. Present the polynomials in the form of a product of two binomials:
a) $(2a + 3b)(a + b) - (a + b)(a - 3b)$;
b) $(2a + 3b)(a - b) - (b - a)(a - 3b)$;
c) $(2x - 5y)(x - y) + (y - x)(x + 3y)$.

12. Find the roots of the equations:

a) $x^2(x + 3) - 4x(x + 3)^2 = 0$; **b)** $x^2(x - 2) + 3x(x - 2)^2 = 0$.

2.2 Factoring. Method of grouping

The method of taking out the common factor is the basis for other methods of factorisation, such as grouping. Let's talk about this method in details.

1. You are looking at a machine that is able to factor by grouping. Examine its operational principle. Use this machine and factor the following polynomial $a^2b - b + ab^2 - a$ на множители.

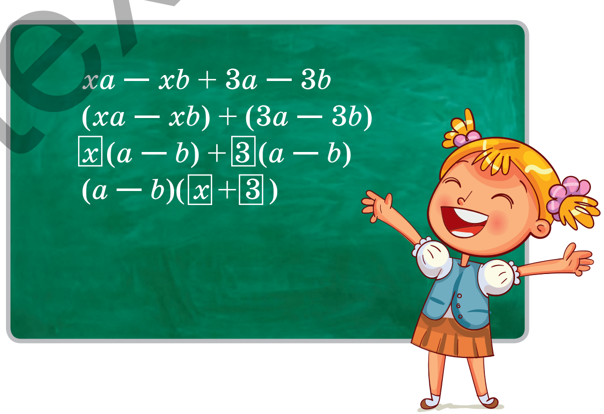
Polynomial	$10a^2 + 15a + 8a + 12$	$a^2b - b + ab^2 - a$
Group the terms ↓	$(10a^2 + 15a) + (8a + 12)$	
Take out the common factor for each group ↓	$5a(2a + 3) + 4(2a + 3)$	
Take out the common factor for each of multiplications ↓	$(2a + 3)(5a + 4)$	
Result	$10a^2 + 15a + 8a + 12 = (2a + 3)(5a + 4)$	

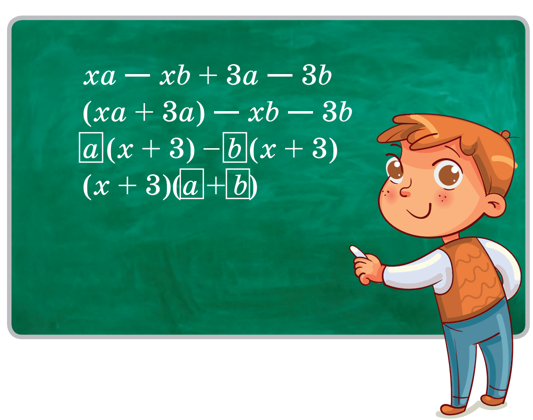
2. Factor the polynomials using the method of grouping:

a) $x(a+b)+4a+4b$; b) $m-n+a(n-m)$;

c) $p(x-y)-4x+4y$; d) $a(b-c)+c-b$.

3. Sabina and Nurlan were solving an example of polynomial factoring. Give comments on their solution. Did they do everything right?

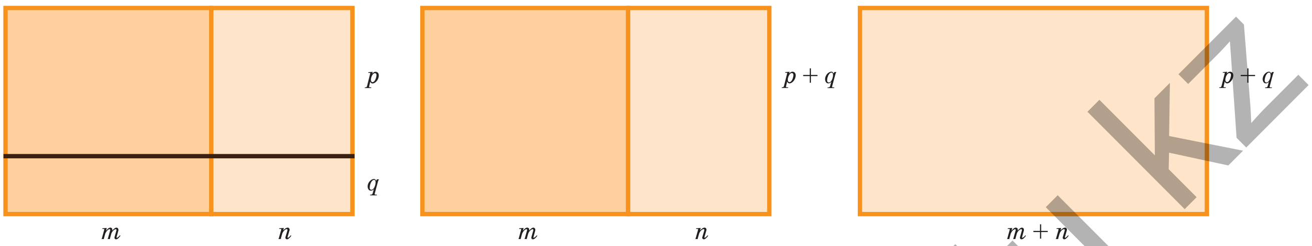




You may factor the same polynomial by grouping different terms..

4. Marat suggested different ways to group the terms of a polynomial when factoring. He presented a geometric interpretation of his method. Give comments on his solution. Is it correct?

$$mp+np+qn+qm=m(p+q)+n(p+q)=(m+n)(p+q)$$



5. Fill the gaps so the polynomial is in the form of a product of several polynomials.

a) $6a-6x+ab-bx=(\square\ldots\square)+(\square\ldots\square)=\square(a-x)+\square(a-x)=\ldots;$

$6a-6x+ab-bx=\ldots=\square(6+b)-\square(6+b)=\ldots;$

b) $x^2+3xy-3x-9y=(\square+\square)-(\square+\square)=\square(x+3y)-\square(x+3y)=\ldots;$

$x^2+3xy-3x-9y=\ldots=\square(x-3)+\square(x-3)=\ldots;$

c) $a^3-a^2b-a+b=(\square\ldots\square)\ldots(\square\ldots\square)=\square(\square\ldots\square)\ldots(\square\ldots\square)=(\square\ldots\square)\ldots(\square\ldots\square).$

6. Factor the polynomials and check the answer by multiplication:

a) $ax-2ay-4bx+8by$; **b)** $12x^3-xy+36x^2y-3y^2$.

7. Factor the polynomials:

a) $ab^2-yb^2-ax+xy+b^2-x$;

b) $ac^2-2ad-bc^2+2cd+2bd-c^3$;

c) $a^4-a^2b^2+b^2x-a^2x-a^2+x$;

d) $a^3y^2-3a^3c+ay^2+y^2-3ac-3c$.

8. Present the polynomials in the form of a product of three factors:

a) $2x^2y-6x^2b-axy+3abx$;

b) $2a^3b-4a^3+ab-2a$;

c) $4x^2y+10xy^2-6ax^2-15axy$;

d) $12a^2b^2-21ab^3-8a^2c+14abc$.

9. Find the roots of the equations:

a) $x^3+3x^2+2x+6=0$; **b)** $x^3+4x^2+5x+20=0$;

c) $x^3-x^2+7x-7=0$; **d)** $x^4-2x^3-8x+16=0$.

10. Factor the polynomial $a^2-ab-3a+3b$ **and find its values, if** $a=3{,}5;\ b=-1{,}7$.

2.3 Factorisation of polynomials. Problem solving

As noted earlier, factorisation allows us to find the best way to solve many mathematical problems. For example, to perform arithmetic calculations quickly, to solve divisibility problems and equations. Let us look at how to apply these methods to solve such problems.

1. Find the values of the expressions:

a) $67{,}3^2 + 67{,}3 \cdot 32{,}7$; b) $37,\ 3\cdot 62,\ 7 + 37,\ 3^2$;

c) $34,\ 2 \cdot 2{,}35 + 2{,}35 \cdot 42{,}3 + 23{,}5 \cdot 2{,}35$; d) $3,7\cdot 6,42 + 7,28\cdot 3,7 - 3,7^2$.

2. Madina is to find the value of the numerical expression given below.

She drew up a plan of the most rational solution and wrote it on cards. Unfortunately, the cards got mixed up. What was Madina's plan? Is it possible to apply it to solutions of other examples? Explain your answer.

a) $\dfrac{1{,}8\cdot 5{,}3 - 1{,}8\cdot 2{,}7}{0{,}4^2 + 0{,}4\cdot 1{,}4} = \dfrac{1{,}8\cdot(\square+\square)}{\square\cdot(\square+\square)}$;

REMEMBER!

Basic property of fractions: the value of a fraction will not change, if the numerator and denominator of the fraction are multiplied or divided by the same number.

1. Divide the numerator and denominator by the common factor
2. Write down the result
3. Take out the common factor in the numerator
4. Find the common factor in the denominator
5. Find the common factor in the numerator
6. Take out the common factor in the denominator

b) $\dfrac{1{,}9\cdot 1{,}6 + 1{,}9^2}{3{,}8\cdot 6{,}7 - 3{,}8\cdot 3{,}2}$; c) $\dfrac{2\frac{3}{7}\cdot\frac{2}{3} - 5\frac{3}{7}\cdot\frac{2}{3}}{\left(1\frac{1}{3}\right)^2 - 1\frac{1}{3}\cdot\frac{1}{3}}$; d) $\dfrac{1\frac{7}{15}\cdot\frac{5}{9} - \frac{5}{9}\cdot\frac{8}{15}}{\left(1\frac{2}{3}\right)^2 - 1\frac{2}{3}\cdot\frac{1}{9}}$;

3. Factor the expressions and find the values:

a) $\dfrac{1{,}6\cdot 7{,}8 - 1{,}6\cdot 5{,}2}{0{,}8^2 + 0{,}8\cdot 1{,}8}$; **b)** $\dfrac{16\cdot 12 - 4\cdot 12 - 16\cdot 2 + 4\cdot 2}{3\cdot 6 + 6\cdot 4 + 5\cdot 6 + 4\cdot 3 + 4\cdot 5 + 4^2}$; **c)** $\dfrac{6{,}1\cdot 3{,}9 - 6{,}1\cdot 1{,}9 - 0{,}4\cdot 3{,}9 + 0{,}4\cdot 1{,}9}{8{,}9\cdot 1{,}7 - 3{,}2\cdot 2{,}3 + 8{,}9\cdot 2{,}3 - 3{,}2\cdot 1{,}7}$.

4. Factor the expressions:

a) $8a^2 + a + a^3 + 8$; **b)** $x^3 + 18 + 3x + 6x^2$;

c) $c^3 - 6 + 2c - 3c^2$; **d)** $18x^2 + 27xy + 14xz + 21yz$.

5. Prove that the value of the expression:

a) $3^9 + 3^8 - 3^7$ is multiple of 11;

b) $3^{n+2} + 4 \cdot 3^n - 9 \cdot 2^n - 2^{n+2}$, $n \in N$, is multiple of 13

c) $6^8 + 6^7 + 6^6$ is multiple of 43

d) $5^{n+1} + 5^{n+2} + 5^{n+3}$, $n \in N$, is multiple of 31

6. Solve the equations:

a) $(2 - x)(x+7) = 0$;

b) $-2x(4x + 1)(5x - 2) = 0$;

c) $5x^2 + 4x = 0$;

d) $x \cdot |x| - 2 \cdot |x| + 4 - 2x = 0$.

7. Arman substituted the polynomial coefficients $ax^3 + bx^2 + cx + d$ with numbers 3, 5, 6, 10, so that the resulting polynomial could be factored. What polynomials did Arman get?

Sometimes, it is more convenient to break up a polynomial term into a sum or difference of like terms to factor the polynomial by grouping.

9. Factor the trinomial by presenting one of its terms as a sum or difference of like terms.

a) $x^2 + 5x + 6$;

c) $b^2 - 2b - 15$;

b) $y^2 + y - 12$;

d) $c^2 - 3bc + 2b^2$.

Example:

$$x^2 + 4x + 3 = x^2 + 3x + x + 3 =$$
$$= x(x+3) + (x+3) = (x+3)(x+1)$$

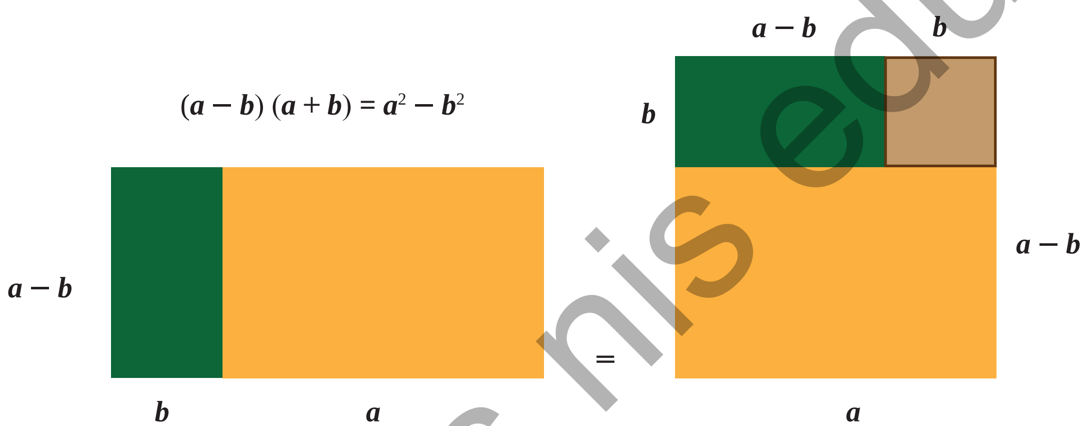
10. Factor the polynomial

$x^{a+b} + x^a y^b - x^b y^a - y^{a+b}$.

2.4 Product of a sum of two expressions by their difference. Difference of squares

You already know that multiplication of a polynomial by another polynomial requires the multiplication of each term of the first polynomial by each term of another polynomial, and the addition of results. This is a long procedure that is not always convenient. In some situations, we can simplify it by using special formulas - the formulas of abridged multiplication.

1. Damir proposed a geometric interpretation of the multiplication of a polynomial ($a - b$) or ($a + b$). Give comments on his solution. Are Damir's conclusions correct?



Conclusion: The product of a sum and difference of two algebraic expressions is equal to the difference of squares of these algebraic expressions, i.e. $(a - b)(a + b) = a^2 - b^2$.

2. Multiply the polynomials:

a) $(a-2b)(a+2b)$; **b)** $(2a-3b)(2a+3b)$;

c) $(3x^2-4y^3)(3x^2+4y^3)$; **d)** $(2xy^2-4xy)(2xy^2+4xy)$.

3. Use mathematical language and simplify the expressions:

a) product of a difference $(4a - 8b)$ and sum $(4a + 8b)$;

b) product of a difference $(n^2 - 3m^3)$ and sum $(n^2 + 3m^3)$.

4. Present the expressions in the form of polynomials:

a) $4a(d - a)(a + d)$; **b)** $-3(x + y)(x - y)$;

c) $-2x(3x+8y^2)(8y^2-3x)$; d) $0{,}5a(4b-6a^2)(6a^2+4b)$.

5. Multiply the polynomials:

a) $(x-y)(x+y)(x^2+y^2)$; b) $(a+2)(2-a)(4+a^2)$;

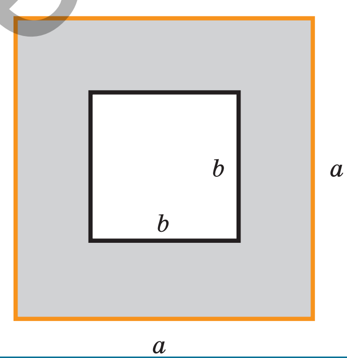
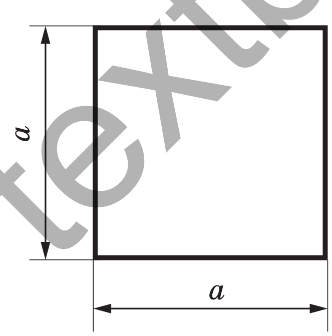
c) $(4+n)(n^2+16)(n-4)$; d) $(1+4b^2)(2b+1)(1-2b)$.

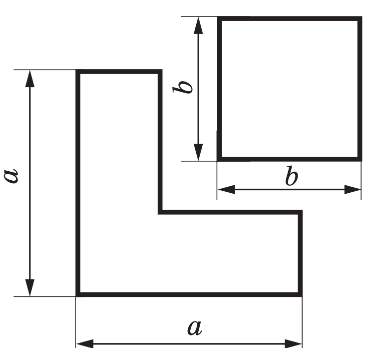
6. Find the values of the expressions:

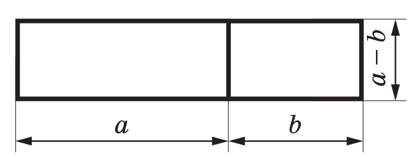
a) $(10-4)(10+4)$; b) $\left(10-\frac{1}{3}\right)\left(10+\frac{1}{3}\right)$;

c) $103\cdot 97$; d) $7{,}8\cdot 8{,}2$;

e) $5\frac{3}{4}\cdot 6\frac{1}{4}$; f) $2{,}7\cdot 3{,}3$.

<table>
<tr><td colspan="2">7. Damir presented a geometric solution to the problem on finding a figure shown in Figure. Give comments on his solution. Are Damir's conclusions correct?</td><td>
</td></tr>
<tr><td>Damir cut a square from a sheet of paper with side a cm.</td><td>Then, he cut another square from this square with side b cm.</td><td>Damir cut the rest of the sheet and folded it into a rectangle with sides (a − b) and (a + b).</td></tr>
<tr><td>

Area: $S=\square$ sm².</td><td>

The area of the square with side b cm is $S=\square$ sm².
The area of the rest part is:</td><td>

The area of obtained rectangle is:
$S=$ ☐</td></tr>
</table>

Conclusion: The difference of squares of two algebraic expressions is equal to the product of a difference of these expressions by their sum, i.e. $a^2-b^2=(a-b)(a+b)$ **.**

REMEMBER!

Formula for the difference of squares of two expressions
$a^2-b^2=(a-b)(a+b)$.

8. Fill the gaps so that it is possible to factor a polynomial by using the formula for the difference of squares.

a) $25a^2-4=(5a)^2-(\square)^2=(5a-\square)(5a+\square)$;

b) $16m^2-81n^2=(\square)^2-(\square)^2=(\square-\square)(\square+\square)$;

c) $49a^4b^2-\square=(\square)^2-(3b^2)^2=(\square-3b^2)(\square+3b^2)$;

d) $(4x+5)^2-(4x-3)^2=((4x+5)-(4x-3))(\square+\square)=$;

e) $(5a+3b)^2-(4b-6a)^2=((\square)-(\square))(\square+\square)=$

9. Present the polynomial as multiplication:

a) $(2a^2+5)^2-25a^2$;

b) $16x^6-(3x^2-2y)^2$;

c) $(7b+6a^4)^2-36b^8$;

d) $49n^4-(m+5)^2$.

10. Solve the equation:

a) $(3x-1)(3x+1)-x(9x+2)+6x=0$;

b) $12n+8n(7-2n)=9n^2-(5n+6)(5n-6)$.

2.5 Square of a sum of two expressions. Perfect square formula

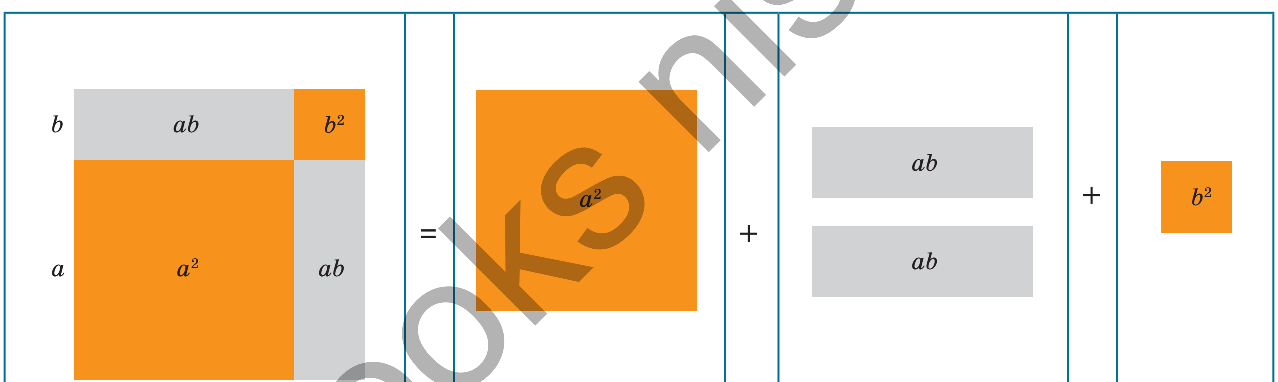
We continue discussing the formulas that allow us to multiply polynomials much quicker than the general rule, so let us consider multiplying the following polynomials $(a + b)$ by $(a + b)$.

1. Translate into mathematical language and simplify:

a) Write the square of a sum of expressions a and b;
b) Write the sum of squares of expressions a and b;
c) Write the twice the product of expressions a and b;
d) Find the square of a sum of expressions a and b by using a definition of powers.

Is it possible to read the obtained expressions as follows:
The square of a sum of two expressions is equal to the square of the first expression plus twice the product of the first expression by the second expression plus the square of the second expression? Why? Explain your answer.

2. Gaukhar presented a geometric illustration of how to derive the formula to find the value of the polynomial $(a+b)^2$. Comment on her solution.



$(a+b)^2$	=	a^2	+	$2ab$	+	b^2

The resulting formula is called the square of a sum of two expressions and is one of the formulas of abridged multiplication.

3. Use the formula for the square of a sum and write the expression as a polynomial:

a) $(x+y)^2$;
b) $(1+b)^2$;
c) $(k+0,5)^2$;
d) $\left(c+\frac{1}{7}\right)^2$;
e) $(2x+1)^2$;
f) $\left(\frac{1}{4}k+m\right)^2$.

REMEMBER!

The formula for the perfect square of a sum of two expressions
$a^2+2ab+b^2=(a+b)^2$.
For example,
$b^2+12b+36=(b+6)^2$.

4. Raise the expression to the quare:

a) $(4x+5y)^2$; b) $(7a^3+8b^2)^2$;

c) $\left(\frac{1}{3}a^2+1\frac{1}{2}ab\right)^2$; d) $\left(2c^3+\frac{1}{4}a^2\right)^2$.

Example:

$(1+3p^2)^2=1+2\cdot1\cdot3p^2+(3p^2)^2=$
$=1+6p^2+9p^4$

Do not forget to use the properties of powers when applying the formulas of abridged multiplication.

5. Fill the gaps to obtain a right equality:

a) $(2+\square)^2=\square+4b+b^2$; **b)** $(k+\square)^2=k^2+\square k+64$;

c) $(6+c)^2=36+\square c+c^2$; **d)** $(a+\square)^2=a^2+10a+\square$.

REMEMBER!

Formula for the perfect square of a sum of two expressions
$a^2+2ab+b^2=(a+b)^2$.
For example,
$b^2+12b+36=(b+6)^2$.

6. Use the formula for the square of a sum to find the square of the numbers:

Example:

$1004^2=(1000+4)^2=1000^2+2\cdot1000\cdot4+4^2=1000000+8000+16=1008016$

a) $(103)^2$; **b)** $(-201)^2$;

c) $\left(4\frac{3}{4}\right)^2$; **d)** $10{,}01^2$.

Sometimes, when solving problems, the square of a sum may be used as $a^2+2ab+b^2=(a+b)^2$. This will help us to write the expression as the perfect square of a sum.

7. Which of these expressions can be represented as a perfect square? Emphasise the features of such expressions.

a) x^2+6x+9; b) $4c^2+25+40c$;

c) $49b^2+14bc+c^2$; d) $16+56m+49m^2$;

e) $81a^2+9b^2+27ab$; f) $8mn+m^2+4n^2$.

8. Factor the polynomial and fill the gaps using the formulas for the difference of squares and the square of a sum of two expressions:

a) $\left(x^2+4x+4\right)-9=\left(x+2\right)^2-\left(\square\right)^2=\left(\left(x+2\right)-\square\right)\left(\left(x+2\right)+\square\right)=\ldots$;

b) $81m^2-\left(4n^2+4n+1\right)=\left(\square\right)^2-\left(\square\right)^2=\left(\square-\square\right)\left(\square+\square\right)$;

c) $49a^4+14a^2b+b^2-\square=\left(\square\right)^2-\left(3b^2\right)^2=\left(\square-3b^2\right)\left(\square+3b^2\right)=$;

d) $\left(9m^2+6m+1\right)-\left(1+2m+m^2\right)=\left(\square+\square\right)^2-\left(\square+\square\right)^2=\ldots$;

e) $\frac{a^2}{9}+\frac{ac}{3}+\frac{c^2}{4}-a^2-2ac-c^2$.

9. Solve the equation:

a) $8x\left(1+2x\right)-\left(4x+5\right)\left(4x-5\right)=3x$;

b) $x-4x\left(1-4x\right)=11-\left(3-4x\right)\left(3+4x\right)$;

c) $\left(x+6\right)^2-x\left(x-8\right)=2$;

d) $\left(2x+3\right)^2-4\left(x-1\right)\left(x+1\right)=49$.

2.6 Square of a difference of two expressions. Perfect square formula

Now you know the formula for the square of a sum of two expressions, so the question is - how to find the square of a difference of two expressions.

As you remember, we have learned how to represent the expression $a-b$ as $a+(-b)$ when studying polynomials. Let's take advantage of this fact and derive a formula to find the difference of squares of two expressions.

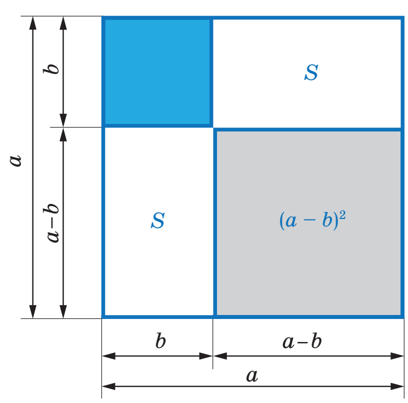
1. Darya has presented two methods to derive the formula $(a-b)^2$. Give your comments. What methods did Darya use?

Method 1	**Method 2**
$(a-b)^2=(a-b)(a-b)=$ $=a^2-ab-ab+b^2=$ $=a^2-2ab+b^2$	$(a+(-b))^2=$ $=a^2+(-2ab)+(-b)^2=$ $=a^2-2ab+b^2$

The square of a difference of two expressions is equal to the square of the first expression minus twice the product of the first expression by the second expression plus the square of the second expression.

2. Use the Figure and prove that

$(a-b)^2=a^2-2ab+b^2$.



3. Use the formula for the square of a difference and write the expression as a polynomial.

a) $(2-x)^2$; b) $(5x-y)^2$; c) $(7c-2b)^2$;

d) $\left(4a-\frac{1}{2}\right)^2$; e) $\left(-m-\frac{2}{3}\right)^2$; f) $\left(-\frac{1}{6}m+3n\right)^2$.

4. The cards have expressions given on them. Which of these expressions are equal? Explain your answer. What patterns did you notice?

$(-a+b)^2$	$(-a-b)^2$	$(a-b)^2$	$(b+a)^2$	$(b-a)^2$

5. Find the squares of numbers using the formula for the square of a difference:

a) $(90-1)^2$; b) 97^2; c) $\left(1\frac{2}{3}\right)^2$; d) $(9{,}98)^2$.

As with the formula for the square of a sum, the formula for the square of a difference can be used in the form of $a^2-2ab+b^2=(a-b)^2$. In this case we are talking about a perfect square of a difference.

REMEMBER!

Formula for the perfect square of a difference of two expressions

$a^2-2ab+b^2=(a-b)^2$.

For example,

$x^2-10x+25=(x-5)^2$.

6. Represent a trinomial as a square of a binomial:

a) $9-6b+b^2$; b) $81m^2+16n^2+72mn$; c) $100b^2+9a^2-60ab$;

d) y^2+25x^2-10xy; e) $48a^2b^3+36a^4b^2+16b^4$; f) $49x^4y^2-42x^4y^3+9x^4y^4$.

7. Complete the table with use of the formula for the square of a sum and difference of two expressions.

1st expression	2nd expression	Square of a sum of two expressions equals	Square of a difference of two expressions equals
$\frac{2}{3}a^3c$	$\frac{1}{4}c^3$		
$-\frac{2}{7}xy$	$-\frac{7}{12}x$		
$-3a^3b$		$9a^6b^2-6a^4b+a^2$	
	$-3m^2n^5$		$4m^2n^6+12m^3n^8+9m^4n^{10}$
$-\frac{2}{5}x^4$			$25y^2-4x^4y+\frac{4}{25}x^8$

8. Represent the expressions in the form of a standard polynomial:

a) $(6x+5y)^2-8(2x-3y)^2$; b) $(7+2ab)(2ab-7)+(6-5ab)^2$; c) $(5-7a)^2-(5a-3)(4-3a)$;

d) $(2x+3)^2-4(5-6x)^2$; e) $(a+b)^2-(b+c)^2+(a+c)^2$; f) $5x(3x+4y)^2-2y(3y+4x)^2$.

9. Kanat made some mistakes when expanding the paraphrases of a sum and a difference of two expressions. Find the mistakes he made.

a) $(2a+b)^2=4a^2+b^2$; b) $(3x-y)^2=9x^2+3xy-y^2$;

c) $(a-2b)^2=a^2-4ab-4b^2$; d) $(c-13)^2=c^2-13c+169$.

10. Represent an expression in the form of A^2+c, where A is a binomial, and c is a number. For example:

a) $a^2+8a+10$; b) $b^2-16b-1$;

c) $y^2+14y+24$; d) $m^2+20m+48$;

Example:

$x^2+6x+2=$

$=(x^2+6x+9)-9+2=$

$=(x+3)^2-7=$

$=(x+3)^2+(-7)$

11. Is an equality true?

$a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc=(a+b)(b+c)(c+a)$.

2.7 Square of a sum and difference of two expressions. Problem solving

With your knowledge of how to expend the square of the sum and difference of two expressions, the difference of squares, and ability to perform arithmetic operations with polynomials and identify perfect squares, great opportunities have been opened to you in solving mathematical problems of all kinds.

1. There are three envelopes and nine cards in front of you. Put the appropriate card in each envelope. Explain your answer:

Square of a sum	Square of a difference	Difference of squares
$(a+b)^2=a^2+2ab+b^2$	$(a-b)^2=a^2-2ab+b^2$	$a^2-b^2=(a-b)(a+b)$

2. Fill the missing monomials so that the equality is correct:

$(-x-y)^2$ $(-x-y)^2$ $(x-y)^2$ $-(x+y)^2$

x^2-y^2 $(-x+y)^2$ y^2-x^2 $(y+x)^2$ $x^2+(-(-y)^2)$

a) $(4a+\square)^2=\square+40ab+\square$; **b)** $(7x-\square)^2=\square-\square+81y^2$;

c) $(\square+8c)^2=\square-80ac+\square$; **d)** $(m^2-\square)^2=\square-50m^2n+\square$;

e) $(\square-\square)^2=81x^6-\square+36y^4$; **f)** $(\square+\square)^2=25p^4+80p^2k^2+\square$.

3. Calculate with use of the formulas of abridged multiplication:

a) $\left(14\frac{1}{14}\right)^2$; $\left(-6\frac{5}{6}\right)^2$; $\left(8\frac{3}{16}\right)^2$; $\left(12\frac{12}{13}\right)^2$;

b) $78\cdot 82$; $31\cdot 29$; $10\frac{1}{8}\cdot 9\frac{7}{8}$; $9\frac{2}{5}\cdot 8,6$.

4. Write the expressions in the form of the sum of squares of two expressions:

a) $a^2+b^2-10b+25$; **b)** x^2+y^2-6y+9;

c) $16m^2+26n^2-40mn$; **d)** $48a^2b^3+37a^4b^2+16b^4$;

e) $a^2-2ac+2c^2+14c+49$; **f)** $10n^2+6nm+m^2-8n+16$.

5. Change one of the coefficients in the expression $36x^2 - 4xy + y^2$ **so that the trinomial could be presented as a square of a binomial. How many ways are there to do this? Explain your answer.**

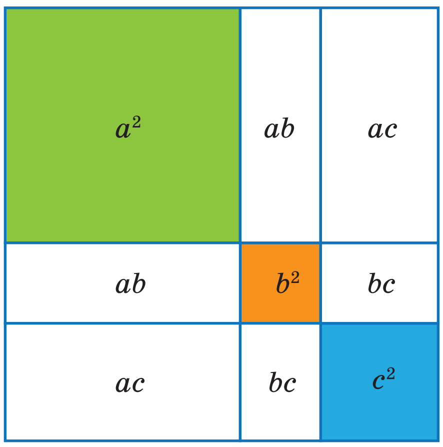
6. It is known that $x + y = 9$ **and** $xy = -15$**. Find the value of** $x^2 + y^2$**.**

7. Make a geometric illustration of this equality.

$(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$

8. Use the Figure below and prove that the equality is correct for positive values of *a*, *b* and *c*.

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.



9. Are the following equalities correct? Why? Explain you answer.

a) $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$;

b) $(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$;

c) $(a-b-c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

2.8 Cube of a sum and cube of a difference of two expressions

Just as we squared the sum or difference of two expressions before, we can cube the sum or difference of two expressions. Let us talk about it in more detail.

> **REMEMBER!**
>
> Formula for the cube of a sum and difference of two expressions
>
> $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
>
> $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

1. Translate into mathematical language and perform the transformations.

a) The cube of a sum of expressions a and b;
b) The sum of cubes of expressions a and b;
c) Triple the product of the square of the expression a by expression b;
d) Use the definition of power and the formula for the square of a sum to find the cube of the sum of the expressions a and b.
e) Compare your results with the solution given below:
Is it true that the obtained expression can be read as follows:

$$(a+b)^3 = (a+b)(a+b)(a+b) = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b) = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 =$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

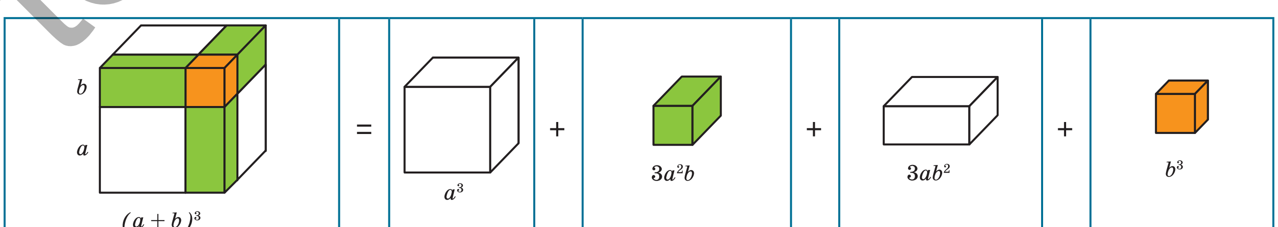
the cube of a sum of two expressions is equal to the cube of the first expression, plus triple the product of the square of the first expression by the second expression, plus triple the product of the first expression by the square of the second expression, plus the cube of the second expression?

e) Use the definition of power and the formula for the square of a difference to find the cube of the difference of the expressions a and b.
Is it true that the obtained expression can be read as follows:
the cube of a difference of two expressions is equal to the cube of the first expression, minus triple the product of the square of the first expression by the second expression, plus triple the product of the first expression by the square of the second expression, minus the cube of the second expression?

2. Use the formula for the cube of a sum and cube of a difference of two expressions, and transform the expression into a polynomial:

a) $(m+n)^3$; **b)** $(m-n)^3$; **c)** $(a+2)^3$; **d)** $(1-p)^3$; **e)** $\left(b-\frac{1}{2}\right)^3$.

3. Gaukhar presented a geometrical illustration of the formula derivation to find the value of the polynomial $(a+b)^3$. Comment on her solution.



$(a+b)^3$	=	a^3	+	$3a^2b$	+	$3ab^2$	+	b^3

Can you propose a geometrical illustration of the formula $(a-b)^3$?

4. Fill the gaps to obtain the correct equality:

a) $(2+\square)^3=\square+3\cdot2^2\cdot\square+3\cdot2\cdot\square^2+b^3$; **b)** $(a+\square)^3=a^3+3\cdot\square\cdot\square+3\cdot\square\cdot\square+64$;

c) $(5-b)^3=\square^3-\square\cdot5^2\cdot b+\square\cdot5\cdot\square-b^3$; **d)** $(x-\square)^3=x^3-3\cdot5\cdot x^2+3\cdot\square\cdot x-\square$.

5. Represent the expression in the form of a polynomial:

a) $(3x+2y)^3$; b) $(3a-2b)^3$; c) $(3x^2-4y^3)^3$; d) $(-m-n)^3$; e) $(-3+p)^3$;

f) $(2k-3p)^3$; g) $(a^2+2b)^3$; h) $\left(-\frac{1}{3}m^2-\frac{3}{2}mn\right)^3$; i) $\left(3a^3-\frac{1}{3}b^2\right)^3$.

6. Find the mistakes of Aliya when cubing a polynomial:

$$\left(\frac{1}{3}x-\frac{3}{4}y\right)^3=\frac{1}{9}x^3-\frac{1}{4}x^2y+\frac{3}{16}xy^2-\frac{9}{16}y^3;$$

$(2a+3b)^3=8a^3+24a^2b+27ab+9b^2$; $(3c^2+2c^4d^3)^3=c^2(27+12c^2d^6+54c^4d^3+27c^6d^6)$

7. Simplify the expression:

a) $(2c+n)^3-6cn(2c+n)$;

b) $(a-m)^3-(a-m)(a^2+am+m^2)$;

c) $(a+b)(a^2-ab+b^2)-(a+b)^3$.

8. Prove that the equality is correct:

a) $(a+b)^3-a^3-b^3=3ab(a+b)$; b) $(a-b)^3=a^3-b^3-3ab(a-b)$.

Sometimes, when solving problems, the formula for the cube of a sum or difference may be used in the following form: $a^3+3a^2b+3ab^2+b^3=(a+b)^3$ и $a^3-3a^2b+3ab^2-b^3=(a-b)^3$. This will help us to write the expression in the form of a so-called cube of a sum or difference of two expressions.

Which of the following expressions can be presented in the form of a cube of a binomial?

a^3-3a^2-3a+1; $b^3+9b^2+9a+27$;

$8p^3+12p^2q+6pq^2+q^3$; b^3+3b^2-3a-1.

REMEMBER!

Formula for the cube of a sum and difference of two expressions:

$a^3+3a^2b+3ab^2+b^3=(a+b)^3$,

$a^3-3a^2b+3ab^2-b^3=(a-b)^3$.

For example,

$b^3+3b^2+3b+1=(b+1)^3$.

2.9 Formulas for the cube of a sum and cube of a difference of two expressions. Problem solving

1. Complete the table:

Name of the formula of abridged multiplication	Writing the formula of abridged multiplication	First expression	Second expression	Result
Square of a difference		$3a^2$	$\frac{1}{2}b^4$	$9a^4-3a^2b^4+\frac{1}{4}b^8$
Difference of squares		$\frac{1}{3}x^4$	$\frac{1}{4}y^2$	
Cube of a sum	$\left(4m^2+\frac{1}{3}n\right)^3$			
Cube of a difference				$8a^9-12a^6b^6+6a^3b^{12}-b^{18}$
Product of a sum and difference of two expressions	$\left(\frac{1}{2}c^3-\frac{1}{3}d\right)\left(\frac{1}{2}c^3+\frac{1}{3}d\right)$			
Square of a sum		$\frac{1}{3}a^3$	$3b^2$	

2. Match:

1.	$(2+b)^3$	**6.**	$27a^6-9a^4b+a^2b^2-\frac{1}{27}b^3$
2.	$\frac{1}{4}a^2+2ab+4b^2$	**7.**	$\left(\frac{1}{3}a+b\right)^3$
3.	$\left(3a^2-\frac{1}{3}b\right)^3$	**8.**	$\left(-\frac{1}{2}a-2b\right)^2$
4.	$2ab-\frac{1}{4}a^2-4b^2$	**9.**	$8+12b+6b^2+b^3$
5.	$\frac{1}{27}a^3+\frac{1}{3}a^2b+ab^2+b^3$	**10.**	$-\left(\frac{1}{2}a-2b\right)^2$

3. Use the formula for the cube of a sum or difference of two expressions and calculate:

a) 31^3; b) 29^3; c) $11{,}1^3$; d) $10{,}9^3$.

4. Find the difference between the cube of a sum of two expressions a and b, and the sum of cubes of the same expressions. Find the difference, if a = -1, b = 1.

5. Find a number to be added to the following expression $64a^3 - 36(4a^2 - 3a)$**, to obtain the sum that is equal to the cube of a binomial.**

6. Solve the equation $(2 + x)^3 = 6x^2 + x^3 - 28$.

7. Fill the gaps.

a) $(\square - 2)^{\square} = b^5 - 6b^4 + 12b^2 - 8$;

b) $(a^3 + \square)^{\square} = a^9 + 3a^7b + 3a^5b^2 + a^3b^3$;

c) $(3a^2 + \square)^{\square} = 27a^6 + 54a^5 + 36a^4 + 8a^3$.

8. Find the value of a numerical expression using the most rational method

$$\frac{0{,}6^3 - 3 \cdot 0{,}6^2 \cdot 0{,}1 + 0{,}03 \cdot 0{,}6 - 0{,}001}{0{,}6 \cdot 0{,}5 - 0{,}5 \cdot 0{,}1}.$$

9. It is known that $x^2 - y^2 = 5$ and $x - y = 0{,}5$**. Find the values of the expressions:**

a) $x + y$;

b) $x^2 - 2xy + y^2$;

c) $x^2 - 2xy + y^2 + 1{,}5$;

d) $x^2 + 2xy + y^2 - x - y$;

e) $x^3 + y^3 + 3x^2y + 14 + 3xy^2$;

f) $x^3 - 3x^2y - y^3 - 15 + 3xy^2 + x - y$.

2.10 Formulas for the sum and difference of cubes of two terms

You have seen that the formulas of abridged multiplication allow us to solve mathematical problems more compactly. You have noticed that in some cases, these formulas allow us to represent a polynomial in the form of the product of two factors, for example: $a^2 - b^2$.

Is there a formula that allow us to factor such expressions with a variable raised to the third power?

Perfect square of a sum or difference of two terms.

$a^2 + 2ab + b^2$
and $a^2 - 2ab + b^2$.

$a^2 - ab + b^2$
and $a^2 - ab + b^2$

Non-perfect square of a sum and difference of two terms.

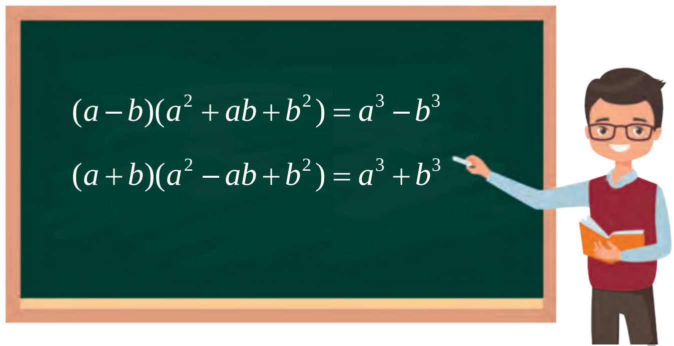
What do you think is a non-perfect square of a sum or difference of two terms? Why is it called so?

What is the difference between a perfect square of a sum or difference and non-perfect square of a sum or difference?

2. Complete the table:

First term	Second term	Non-perfect square of difference	Non-perfect square of difference	Non-perfect square of sum	Perfect square of sum
x	y	$x^2 - xy + y^2$	$x^2 - 2xy + y^2$	$x^2 + xy + y^2$	$x^2 + 2xy + y^2$
$2m$	$3n$				
a^2	$3b^2$				
$5p$					$25p^2 + 40pq + 16q^2$
	2	$y^2 + 2y + 4$			
			$9k^2 - 30k + 25$		
				$q^4 + q^2p^3 + p^6$	

3. Kanat found the product of the binomial $(a-b)$ by the trinomial (a^2+ab+b^2) and the binomial $(a+b)$ by the trinomial (a^2-ab+b^2), and wrote down the results on a blackboard. Are they correct? Why? Explain your answer. How can you read the formulas derived by Kanat?



4. Write down the product as a polynomial:

a) $(a+1)(a^2-a+1)$;
b) $(b-4)(b^2+4b+1)$;
c) $(1+2m)(1-2m+4m^2)$;
d) $(3-5p)(9+15p+25p^2)$;
e) $(m^2-1)(m^4+m^2+1)$;
f) $(a^2-b^2)(a^4+b^4+a^2b^2)$;
g) $(2a+3b)(4a^2+9b^2-6ab)$;
h) $(5m^3+2)(25m^6+4-10m^3)$;
i) $\left(\frac{2}{5}x^3-\frac{1}{3}y^2\right)\left(\frac{4}{25}x^6+\frac{2}{15}x^3y^2+\frac{1}{9}y^4\right)$
j) $(0{,}5a^4+0{,}3b^4)(0{,}25a^8-0{,}15a^4b^4+0{,}09b^8)$.

5. Fill the gaps, so the equality is correct:

a) $(3x+\square)\,(9x^2-3x\square+\square^2)=27x^3+64x^3$;
b) $(\square-5b)(\square^2+5b\square+25b^2) = a^6-125b^3$;
c) $(-\square-3n)(\square^2-3n\square+9n^2)=-27n^3-64m^6$;
d) $(-4b+\square)(16b^2+4b\square+\square^2)=(125a^9-64b^3)$;
e) $(4a+\square)(16a^2-4a\square+\square^2)=64a^3+125b^9$;
f) $(\square-6b)(\square^2+6b\square+36b^2)=27a^6-216b^3$.

6. Substitute *A, B, C, D* with polynomials, so the equality is correct.

a) $(3a+A)\,(B+16b^2)=C^3-D^3$;
b) $(A-3n)(25m^2-B)=C^3+D^3$;
c) $(3x+A)(B+C)=y^9+D^3$;
d) $(5z-A)(B-C)=D^3-64t^{12}$;
e) $(2m-A)(B+9n^2)=C^3+D^3$;
f) $(A+5b)(B+C)=a^{12}-D^3$.

7. Simplify the expression and find its value. Use various formulas of abridged multiplication.

a) $x(x+3)(x-3)-(x-2)(x^2+2x+4)$ if $x=-\frac{2}{9}$;
b) $2x^3-27-(x+3)(x^2-6x+9)$ if $x=0{,}5$;

8. Solve the equation:

a) $(x-2)(x^2+2x+4)-x(x+3)(x-3)=28$;
b) $(2x-1)(4x^2+2x+1)-2x(2x+3)(2x-3)=53$;
c) $(3x-5)(9x^2+15x+25)-3x(3x+4)(3x-4)=3x+10$;
d) $(x+3)(x^2-3x+9)-x(x+5)(x-5)=-23$;
e) $(4x-5)(16x^2+20x+25)-4x(4x+3)(4x-3)=12x+11$.

2.11 Formulas for a sum and difference of cubes of two expressions. Problem solving

We will contrinue discussing the formulas for a sum and difference of two expressions and pay attention to the following form $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. As can be seen, this will help us to factor a polynomial.

1. Factor a polynomial using the formula for a sum or difference of cubes.

a) $1 - a^3$; b) $n^3 + 27$; c) $-m^3 + 8$;
d) $p^3 + q^3$ e) $8a^3 - 125b^3$; f) $64 + 27a^3$;
g) $a^6 + b^6$; h) $m^3 - n^6$; i) $\frac{a^3}{27} - \frac{b^6}{8}$;
j) $\frac{27}{64}a^6 + \frac{64}{125}b^9$; k) $\frac{10}{27}x^9 - 3\frac{3}{8}y^6$ l) $0{,}008a^3 - 0{,}000001b^3$.

2. Fill the gaps, so the equality is correct:

a) $a^3 + 27 = (a + 3)(\square - 3a + \square)$; b) $\square - b^3 = (\square - \square)(16 + 4b + \square)$;

c) $\square + 64m^9 = (2n^2 + \square)(\square - 8n^2m^3 + \square)$; d) $125p^3 - \square = (\square - 3q^4)(\square + \square + \square)$.

3. Use the formulas for a sum and difference of cubes to prove that the following expression:

a) $36^3 + 14^3$ is divided by 125000; b) $14^3 - 12^3$ is divided by 508.
c) $225^3 + 85^3$ is divided by 31; d) $7102^3 - 5025^3$ multiple of 2017;
e) $321^3 + 179^3$ is divided by 500; f) $743^3 - 543^3$ is divided by 200.

4. Use the formulas of abridged multiplication and calculate:

$\frac{42^3 - 28^3}{14} + 42 \cdot 28$; б) $\frac{49^3 + 31^3}{80} - (49^2 + 31^2)$.

5. Arman wrote the expression $a^6 - b^6$ and factored it using different methods. Examine and comment on these factorisation methods.
Represent the expression $a^{12} - b^{12}$ in the form of the product of a polynomial.

Expression	$a^6 - b^6$	$a^{12} - b^{12}$
1. Represent this expression as a difference of cubes	$(a^2)^3 - (b^2)^3$	
2. Use the formula for the difference of cubes	$(a^2 - b^2)(a^4 + a^2b^2 + b^4)$	
3. Use the formula for the difference of squares and factor the polynomial	$(a - b)(a + b)(a^4 + a^2b^2 + b^4)$	

Expression	$a^6 - b^6$	$a^{12} - b^{12}$
1. Represent this expression as a difference of squares	$(a^3)^2 - (b^3)^2$	
2. Use the formula for the difference of squares	$(a^3 - b^3)(a^3 + b^3)$	
3. Use the formula for the difference and sum of cubes and factor the polynomial	$(a - b)(a + b)(a^2 + ab + b^2)(a^2 + ab + b^2)$	

6. Represent the expressions in the form of the product:

a) $(x + 2)^3 + x^3$; b) $(x + y)^3 - (x - y)^3$;

c) $(2a + b)^3 - (a - 2b)^3$; d) $(3ab - 1)^3 + 1$;

e) $(y - 1)^3 - 27$; f) $(2a - 3b)^3 + 27b^6$.

7. If n has any natural value, prove that the following expressions:

a) $(n + 23)^3 - (n + 5)^3$ is multiple of 18;

b) $(n + 36)^3 + (n - 6)^3$ is multiple of 15;

c) $(n + 3)^3 - (n - 7)^3$ is multiple of 10.

8. Prove that:

a) $a^3 - b^3 = (a - b)^3 + 3ab\,(a - b)$; b) $a^3 + b^3 = (a + b)^3 - 3ab\,(a + b)$;

9. Calculate the value of the expression:

a) $x^3 + y^3$, if $x + y = 4$ and $xy = 5$; b) $x^3 - y^3$, if $x - y = 3$ and $xy = 25$.

10. Factor:

a) $x^{3n} - y^{3n}$; b) $a^{3k} + b^{3k}$; c) $p^{3n-3} - q^{3n-3}$;

d) $z^{3k+3} + t^{3k+3}$; e) $27x^{9k} - 125y^{27k}$; f) $x^{6n} - 64y^{9n}$.

11. Find the difference between the cube of a sum of two expressions a and b and the sum of these cubes. What is the value of the obtained expression, if $a = -\frac{1}{2}$ и $b = -2$?

2.12 Formulas of abridged multiplication. Problem solving

Now you know a sufficient number of formulas of abridged multiplication, so let us consider how to use them in problem solving.

Formulas of abridged multiplication

$$a^2-b^2=(a-b)(a+b);\quad (a+b)^2=a^2+2ab+b^2;\quad (a-b)^2=a^2-2ab+b^2;$$

$$a^3+3a^2b+3ab^2+b^3=(a+b)^3;\quad a^3-3a^2b+3ab^2-b^3=(a-b)^3;$$

$$a^3-b^3=(a-b)(a^2+ab+b^2);\quad a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc;$$

1. Match the left and right parts of the table to find an appropriate 'pair'.

1.	$(x-6)(x+6)$	A.	$(x+2)^3$
2.	$25-10x+x^2$	B.	x^3+64
3.	$(2+x)(4-2x+x^2)$	C.	$(x+4)^2$
4.	$x^2+8x+16$	D.	x^3-1
5.	$x^3+6x^2+12x+8$	E.	$27-x^3$
6.	$(x+4)(x^2-4x+16)$	F.	x^2-36
7.	$(x-1)(x^2+x+1)$	G.	$(5-x)^2$
8.	$27x^3-27x^2+9x-1$	H.	$(3x-1)^3$
9.	$(3-x)(9+3x+x^2)$	I.	$8+x^3$

2. Use the formulas of abridged multiplication and fill the gaps:

а) $a^2-\square=\left(\square-\frac{1}{2}b^2\right)(a+\square)$;   б) $(\square+3y^2)^2=49x^6+\square+\square$;

в) $a^3-\square=(\square-b^2c^3)(a^2\square\square+\square)$;   г) $(2+\square)^3=\square+12x+6x^2+\square$.

3. Simplify and find the value of the expression:

a) $2a^2-28a+98$ if $a=507$;   b) $\dfrac{1}{\frac{1}{2}a^2-6a+18}$ if $a=306$;

c) $64-(4-5x)(16+20x+25x^2)$ if $x=\dfrac{1}{5}$;   d) $4-(2x-3)(4+6x+9x^2)$ if $x=-\dfrac{1}{3}$.

4. Find the roots of the equation:

1) $(3x-7)(3x+7)-(3x-5)^2=16$;   2) $(x+2)(x^2-2x+4)-x(x-4)(x+4)=59$;

3) $8x^3-(2x-1)^3=12x^2-17$.   4) $(x+3)^3-x(x+4)(x-4)-9x^2=156$

5. Calculate the value of the expression using the most rational method:

a) $\dfrac{95^2-27^2}{95^2-2\cdot95\cdot27+27^2}$;

b) $\left(\dfrac{74^3+56^3}{130}-74\cdot56\right):(23^2-5^2)$;

c) $(2^2+1)(2^4+1)(2+1)(2-1)+1$;

d) $3^{32}-8(3^2+1)(3^4+1)(3^8+1)(3^{16}+1)$.

6. Find the value of the expression:

a) $4a^2+9b^2$, if $3b+2a=14$; $ab=5$;

b) x^2+9y^2, if $x+3y=8$; $xy=10$.

c) $bc+ab-ac$, if $a^2+b^2+c^2=45$, $a-b+c=7$

7. Translate the statement into mathematical language and solve the problem:

a) Find three consecutive natural numbers, so that the sum of squares of the first two numbers is equal to the square of the third number.

b) Find consecutive natural numbers, so that the sum of cubes of three consecutive numbers is equal to the cube of the following number.

8. Examine the Pascal's triangle and determine a pattern when finding coefficients of the terms in the expansion of a binomial.

a) Find the expansion coefficients of the polynomial $(a+b)$ with use of the Pascal's triangle, and fill the gaps.

$(a+b)^6=a^6+\dots a^5b+\dots a^4b^2+\dots a^3b^3+\cdots a^2b^4+\cdots ab^5+\cdots+b^6$;

b) Factor the polynomial $(a+b)^8$.

Did you know?

In 1655, Blaise Pascal, a French mathematician, discovered a method to find the coefficients when raising any binomial into any n power. In Mathematics this is called **the Pascal's triangle.**

$(a+b)^0=1$

$(a+b)^1=a+b$

$(a+b)^2=a^2+2ab+b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+6ab^4+b^5$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The Pascal's triangle

2.13 Formulas of abridged multiplication. Problem solving

1. Complete the sentences given on a blackboard:

- Difference of squares of two expressions is...
- Square of a sum of two expressions is...
- Product of the difference of two expressions by their sum...
- Square of a difference of two expressions is...
- Difference of cubes of two expressions is...
- Sum of cubes of two expressions is...
- Product of the difference of two expressions by non-perfect square of their sum is...
- Product of the sum of two expressions by non-perfect square of their difference is...

2. 'Match a pair'. Find equal expressions.

$\frac{1}{25}b^4-c^2$	$(a^2-3)^2$	$\left(\frac{1}{4}a-b\right)\left(\frac{1}{16}a^2+\frac{1}{4}ab+b^2\right)$	$a^9-3a^7b+3a^5b^2-a^3b^3$
$\frac{1}{8}a^6+8a^3$	$(b^2-2)^3$	$(7-x)^2$	$9a^6+4c^4+12a^3c^2$
$\frac{1}{64}a^3-b^3$	$49-14x+x^2$	$\left(\frac{1}{2}a^2+2a\right)\left(\frac{1}{4}a^4-a^3+4a^2\right)$	$(3a^3+2c^2)^2$
$b^6-6b^4+12b^2-8$	a^4+9-6a^2	$\left(\frac{1}{5}b^2-c\right)\left(\frac{1}{5}b^2+c\right)$	$(a^3-b)^3$

3. Solve the following equations:

a) $(x-2)(x^2+2x+4)=0$;

b) $(x+1)(x^2-x+1)=-7$;

c) $(x-1)(x^2+x+1)=7$;

d) $(x+2)(x^2-2x+4)=9$.

4. Compare the values of the expressions without making calculations:

a) $(499+236)^2$ and 499^2+236^2;

b) 154^2+196^2 and $(154+196)^2$;

c) $(783-563)^2$ and 783^2+563^2;

d) 386^2+244^2 and $(386-244)^2$.

5. Knowing that $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$,

a) derive the formula for $(a+b-c)^2$; $(a-b+c)^2$ and $(a-b-c)^2$;

b) find the value of the expression $a^2+b^2+c^2$, if $a+b-c=8$, $ab-bc-ac=5$;

c) find the value of the expression $bc+ab-ac$, if $a^2+b^2+c^2=45$, $a-b+c=7$

6. Prove the identity:

a) $a^4-b^4=(a-b)(a^3+a^2b+ab^2+b^3)$;

b) $a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$;

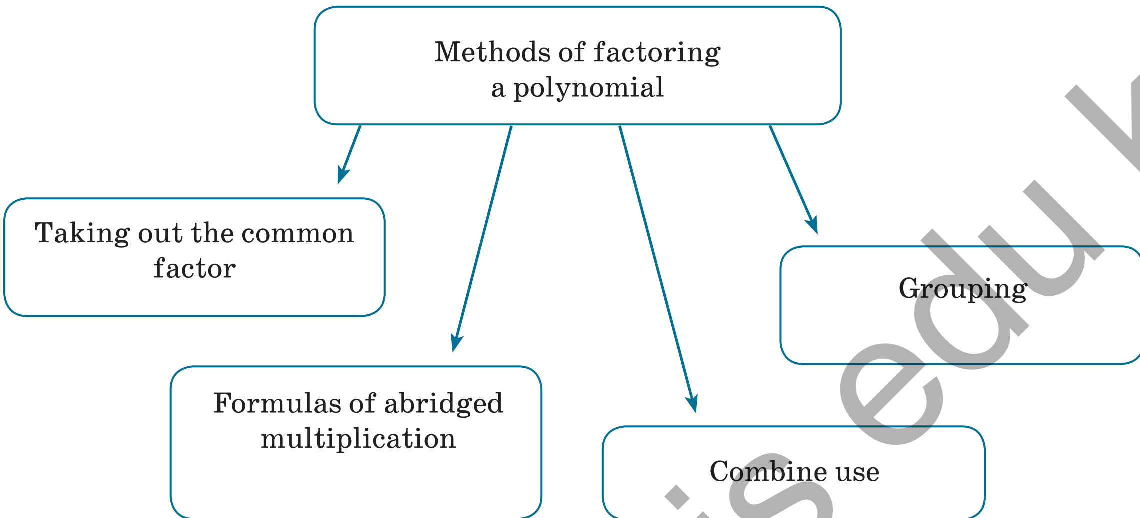
c) $a^4-b^4=(a-b)(a+b)(a^2+b^2)$;

d) $a^5-b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$.

2.14 Different methods of factoring

Previously we have studied factoring. Let us systematise all the methods that you know and use them in problem solving.

1. Examine the methods of factoring a polynomial. Characterise each method.



2. Find a 'divisor' of the polynomial:

$x^2 - y^2$
$x^2y - xy$
$(x-y)^2$
$x^3y + xy^3$
$x^2y + y - 2xy$

$(x+y)$ $\quad$ $x-1$

$(x-y)$ $\quad$ $x+1$

$x^3y - y^3$
$x^2y - y$
$x^2 + 2xy + y^2$
$x^4y - xy^4$
$x^2 + xy$

3. Factor the polynomial. What factoring methods can you use? Explain your answer.

a) $x^2 + xy + xz + yz$;

b) $6p^2 + 4pq + 9pr + 6qr$;

c) $16 - 25a^2b^2$;

d) $a^3 - 9ab^2$;

e) $a^4 - 2a^3 + a^2 - 1$;

f) $b^4 - b^2 - 2b - 1$;

g) $(x+y)^2 - z^2$;

h) $x^2 - (y+z)^2$;

i) $n^2 + 18n + 81 - m^2$;

j) $x^2 - y^2 + z^2 - p^2 + 2xz + 2yp$;

k) $p^3 - 27q^3$;

l) $(x+y)^3 - (2x-y)^3$;

m) $(x-2)^3 + 1$;

n) $1 - (x+2)^3$;

o) $x^3 - y^3 + 3y^2 - 3y + 1$;

p) $8x^3 + y^3 + 6y^2 + 12y + 8$;

4. Is it true that:

a) $929^2 - 71^2$ is divided by 1000;
b) $31^2 + 2 \cdot 31 \cdot 14 + 14^2$ is divided by 45
c) $63^3 + 67^3$ is divided by 60;
d) $847^3 - 222^3$ is divided by 25;
e) $3^4 + 3^5 + 3^6$ is divided by 13;
f) $7^n + 7^{n+1} + 7^{n+2}$ is divided by 57?

5. Prove that for any integer value x, the following expression:

a) $(3x + 1)^2 - (2x - 1)^3$ is divided by 12;
b) $(2x + 1)^3 + (2x - 1)^3$ is divided by 4.

6. Is it true that:

a) difference of squares of two consecutive even numbers is divided by 4;
b) difference of squares of two consecutive odd numbers is divided by 8?

7. Use the most rational method to calculate the value of the following expression:

a) $\dfrac{109^2 - 2 \cdot 109 \cdot 77 + 77^2}{79^2 + 73^2 - 49^2 - 55^2}$;

b) $2017^2 - 2016 \cdot 2018$;

c) $(100^2 + 98^2 + 96^2 + 94^2 + 92^2) - (99^2 + 97^2 + 95^2 + 93^2 + 91^2)$.

8. Marat and Layla play a game. They have cards with monomials written on them. They have to make three polynomials of the following form $ax^4 + bx^3 + cx^2 + dx + e$ so that each monomial could be used only once and this polynomial could be factored.

$6xy$	$3xy$	$2xy$	$-8x$
$-6x$	$-4x$	$3y$	y
$-y$	-4	2	-2

2.15 Different methods of factoring. Problem solving

1. Solve the equations:

a) $y^2(y-4)+2y(y-4)+y-4=0$;

b) $y(y^2-6y+9)-4(y^2-6y+9)=0$;

c) $y^2(y^2-8y+16)-4y(y^2-8y+16)=0$;

d) $y^2(y^2-12y+36)-6y(y^2-12y+36)+9(y^2-12y+36)=0$.

2. Factor the polynomial by representing one of its terms as a sum or difference of monomials:

a) a^4-3a^2+9; b) $7a^2+a-8$; c) $x^2+ax-2a^2$;

d) $x^4+5x^2y^2+6y^4$; e) $x^3+3bx^2-4b^2x$; f) $x^5+x^3y^2+xy^4$.

One of the factoring method is perfect square. Let us talk about this in more detail.

3. Here we have an example of factoring a perfect square. Comment on the factoring and perform the tasks.

Steps	Polynomial 1	Polynomial 2
1. An original polynomial	x^2-8x-9	x^2+4x-5
2. Complete the perfect square Polynomial	$x^2-8x+16-16-9$	
3. Use the formula for the perfect square of a sum or difference of two expressions	$(x-4)^2-25$	
4. Use the formula for the difference of squares of two expressions	$(x-4-5)(x-4+5)$	
5. We will get a factorisation of the polynomial	$(x-9)(x+1)$	

Steps	Polynomial 1	Polynomial 2
1. An original polynomial	$2x^2-x-3$	$2x^2+3x-5$
2. Put the coefficient outside the brackets	$2\left(x^2-\frac{x}{2}-\frac{3}{2}\right)$	
3. Complete the perfect square polynomial	$2\left(x^2-2\cdot\frac{x}{4}+\frac{1}{16}-\frac{1}{16}-\frac{3}{2}\right)$	

4. Use the formula for the perfect square of a sum or difference of two expressions	$2\left((x-\frac{1}{4})^2-\frac{25}{16}\right)$	
5. Use the formula for the difference of squares of two expressions	$2\left(x-\frac{1}{4}-\frac{5}{4}\right)\left(x-\frac{1}{4}+\frac{5}{4}\right)$	
6. We will get a factorisation of the polynomial	$2\left(x-\frac{3}{2}\right)(x+1)=(2x-3)(x+1)$	

Steps	Polynomial 1	Polynomial 2
1. An original polynomial	x^4+16	x^4+64
2. Complete the perfect square Polynomial	$x^4+4x^2+16-4x^2$	
3. Use the formula for the perfect square of a sum or difference of two expressions	$(x^2+4)^2-4x^2$	
4. Use the formula for the difference of squares of two expressions	$(x^2+4-2x)(x^2+4+2x)$	
5. We will get a factorisation of the polynomial	$(x^2-2x+4)(x^2+2x+4)$	

4. Write the algebraic expression in the form of product of polynomials:

a) $x^8+x^4y^4+y^8$;   b) x^2-8x+7;

c) $a(a+2)-(b+1)(b-1)$;   d) $(a+b-2)(a+b)-(a-b)^2+1$.

5. Solve the equation:

a) $x^3+4x^2+5x+2=0$;   b) $x^3-8x^2+13x-6=0$;

c) $x^3-6x^2+3x-1=0$;   d) $x^4-2x^2-400x=9999$ (Bhaskara's equation).

6. Ruslan presented evidences on a blackboard that all the numbers are equal. Are the evidences correct?

All the numbers are equal.
Let us take two random numbers and mark them as x and y, where $x>y$.
Let us assume that $x-y=z$, then $x=y+z$, and multiply this inequality by the expression $(x-y)$, we will get $x(x-y)=(y+z)(x-y)$.
Remove the brackets and take out the common factor
$x^2-xy=xy-y^2+xz-yz$, $x^2-xy-xz=xy-y^2-yz$, $x(x-y-z)=y(x-y-z)$,
I divide both parts of the equality into $(x-y-z)$, and get $x=y$.

2.16 What have I studied?

When you are done, you will repeat what you learned about the formulas of abridged multiplication and the methods of polynomial factoring.

FORMULAS OF ABRIDGED MULTIPLICATION

Formulas of abridged multiplication

- The product of the difference and sum of two expressions is…;
- The difference of squares of two expressions is…;
- The square of the sum of two expressions is…;
- The square of the difference of two expressions is…;
- The cube of the sum of two expressions is…;
- The cube of the difference of two expressions is…;
- The sum of the cubes of two expressions is…;
- The difference of the cubes of two expressions is…;
- The square of the sum of three expressions is….

Factorisation of a polynomial

- Taking out a common factor…
- Factoring by grouping…
- Completing the square…

The list of questions to review the previously studied materials.

Complete the sentences using the following words at least once:

- factorisation of a polynomial;
- taking out a common factor;
- factoring by grouping;
- difference of the squares of two expressions;
- perfect square of a sum;
- perfect square of a difference;
- square of the sum of two expressions;
- square of the difference of two expressions;
- cube of the sum of two expressions;
- cube of the difference of two expressions;
- non-perfect square of a sum;
- non-perfect square of a difference;
- sum of the cubes of two expressions;
- difference of the cubes of two expressions.

1. Take out the common factor:

a) $3a^2 - 5ab + 8ab^2$;
b) $a^{2n} - 2a^n - 4a^{2n+1}$;
c) $m(a-b) - n(a-b)$;
d) $5b(x-y) - 3a\,(y-x)$;
e) $(3x+2y)(a-b) - (b-a)(3x+2y)$;
f) $(m^2-n)(x^3-2x+1) + n(x^3-2x+1)$;
g) $7a(2x-y) - 8b(2x-y) + 9c(y-2x)$;
h) $6a^3b\,(x^2-3x+2) - 8a^2b^2(x^2-3x+2)$.

2. Factor the polynomial by grouping

a) $mx - my + nx - ny$;   b) $mx - nx + m - n$;

c) $8ay-4by+2ax-bx$;   d) $3x^2 - 3bx - 3b + 3x$;

e) $my^2 - ny^2 - my + ny - m + n$;   f) $m^2 - mn + m - mn^2 + n^3 - n^2$.

3. Represent as a polynomial in standard form:

a) $(5x - 3b)^2$; b) $(0{,}3a^2 + 5b^3)^2$;

c) $(2x - 3y)^3$; d) $\left(3x^2 - \frac{1}{3}y^3\right)^3$.

4. Factor a polynomial using different techniques:

a) $36p^2 - 25$;   b) $x^6 - 4$;

c) $1 - 8a^6$;   d) $(a + b)^2 - 25a^2b^2$;

e) $64x^9 - 8y^6$;   f) $(x^2 + y^2)3 + (a^2 + 1)^3$.

g) $a^3 + a^2b - ab^2 - b^3$;   h) $(x + y)^4 - (x - y)^4$.

5. Factor a trinomial and represent its middle term as a sum or difference of monomials:

a) $x^2 +x - 6$;   b) $x^2 + 2x - 24$;   c) $x^2 - 6x + 8$.

6. Calculate the value of the expression by the most rational way:

a) $41 \cdot 39$;   b) $100{,}25 \cdot 99{,}75$;   c) 75^2

7. Prove that the following expression:

a) $62^3 - 45^3$ multiples of... 17; b) $225^2 - 85^2$ multiples of... 17; c) $1302^3 + 715^3$ multiples of... 2017.

8. Is the following equality correct?

a) $(a - b)^2 + 4ab = (a + b)^2$; b) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

9. Find the value of the polynomial, if $x = 3, y = 3$:

a) $(x + y)^2 - (x - y)^2$;

b) $(x + y)^3 - (x - y)^3$;

c) $(x - y)^3 - (x^3 - y^3)$.

10. Solve the equation:

a) $x + (5x + 2)^2 = 25(2 + x^2)$;   b) $x^3 + 27 = (x + 3)^3$.

2.17 What have I learned? Assessment activities

1. Take out the common factor:

a) $(2a+6)^2$; **b)** $(3x-15)^2$; **c)** $(5c+5b)^2$;

d) $(15a+12b)^3$; **e)** $(12-16a)^3$; **f)** $(8-6y)^4$.

2. Factor the expression:

a) $a(x+3)^2+b(x+3)$; **b)** $c^2(a+b)-4(a+b)$;

c) $4(3x-5)-3x(5-3x)$; **d)** $(x-y)^2-x(y-x)$;

e) $a^3+a^2b+a^3b-ab^3-ab^2-b^3$; **f)** $0,9xy+1,2y^2-1,2xz-1,6yz$.

3. Prove that:

a) $24^{24}-24^{23}$ is divisible by 92;

b) $32^{70}+68\cdot 32^{69}$ is divisible by 100;

c) 64^3-14^3 is divisible by 25.

4. Factor the polynomial:

a) $49-a^4-6a^2b^3-9b^6$; b) $2x^2-8xy+8y^2-32x^4$;

c) $a^3+3a^2b+3ab^2+b^3-27$; d) $64+x^3-3x^2y+3xy^2-y^3$.

5. Calculate using the most rational way:

a) $\dfrac{7,6\cdot 3,6-6,7\cdot 7,2+7,6\cdot 6,4-6,7\cdot 2,8}{0,2\cdot 2,3-0,2\cdot 1.3}$; b) $(844+840)^2-4\cdot 844\cdot 840$.

6. Solve the equations:

a) $\dfrac{1}{25}x^2-\dfrac{1}{16}=0$; b) $(4x-3)^2-9=0$; c) $(x-3)^2-x(x-8)=2$; d) $(x+1)(x^2-x+1)=9$.

3 Parallel lines

By the end of this unit, I will have learned:

✔ Conditions and properties of parallel lines;

✔ how much is the sum of triangles;

you will be able to:

✔ determine parallel lines and use their properties to find sums of angles of triangle.

Carl Friedrich Gauss

Janos Boyai

Nikolay Ivanovich Lobachevsky

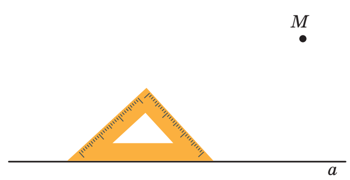
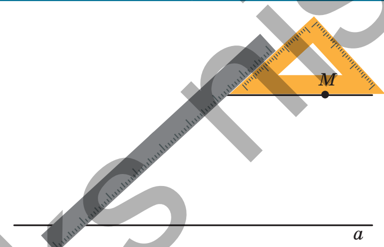
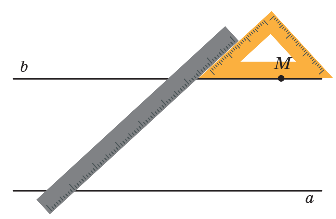
3.1 Parallel lines in the plane

Earlier, we examined the relative position of two lines, and you know that they can intersect, superimpose or have no common points i.e. be parallel. You know how to draw a line perpendicular to a given one.

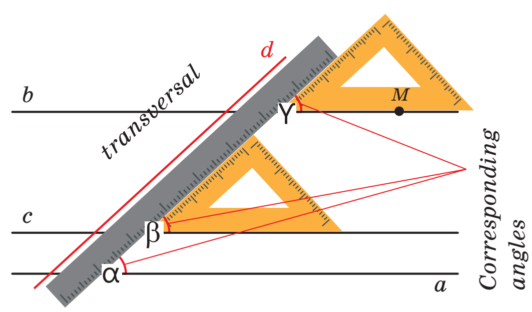
In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.

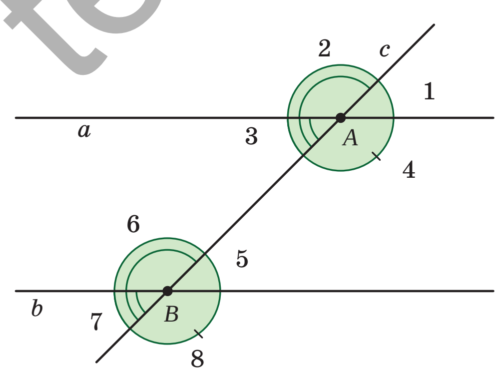
On this lesson we will learn how to construct a line parallel to a given one and passing through given point in the plane. In this case, you can use a property you already know: two lines perpendicular to the third line are parallel.

1. Construct following the plan. Use a ruler and a set square.

Construction: 	**Given:** line a and point $M \notin a$. Construct: line $b \parallel a$, $M \in b$.	
		
1. Position an edge of the set square along the line a, as shown in the drawing.	2. Position a ruler against the set square and move the set square along the ruler until the point M is on the side of the set square.	3. Draw a line b through the point M. Line a is parallel to line b and $M \in b$.

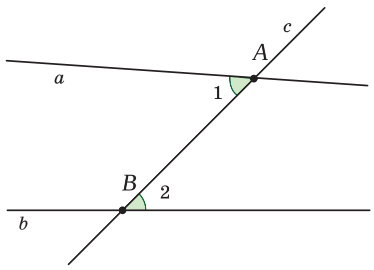
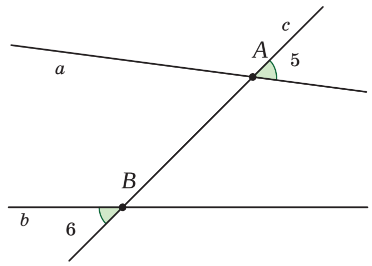
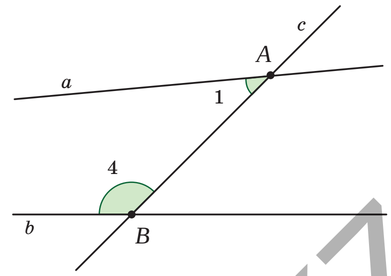
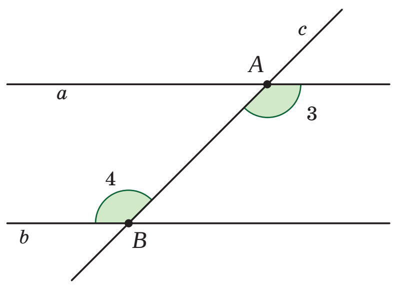
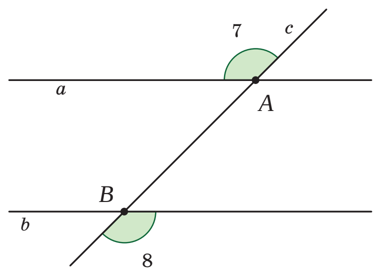
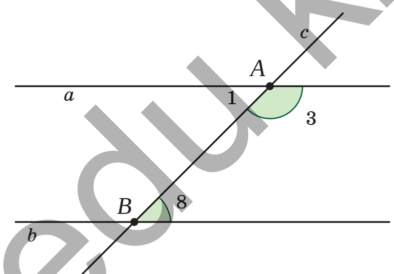
As a result of constructing a line parallel to the given one, angles α, β and γ, were formed. These angles are called corresponding angles for lines a, b, c and line d that intersects these line. This line is called transversal.





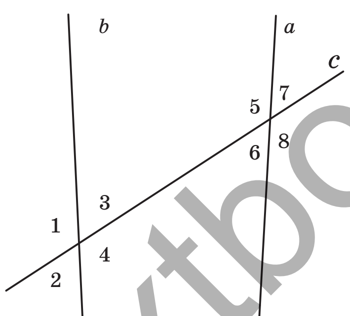
2. Find all pairs of corresponding angles in the drawing. Explain your answer.

Intersection of two lines a and b by a transversal c forms both corresponding and other pairs of angles that also have special names.

Alternate interior angles: 1 and 2, 3 and 4.	Alternate exterior angles: 5 and 6, 7 and 8.	Consecutive angles: 1 and 4, 3 and 8.
		
		

3. Work with the drawing and fill out the table. You may use the properties of adjacent and vertically opposite angles to complete the task.

Given:
$a \cap c$,
$b \cap c$,
$\angle 1 = 106°$, $\angle 8 = 124°$.



Types of angles	Designation of angles	Degree measure
Alternate interior angles		
Alternate exterior angles		
Interior consecutive angles		
Exterior consecutive angles		
Corresponding angles		
Supplementary adjacent angles		
Vertically opposite angles		

4. Complete sentences using the drawing.

a) Consecutive angles for lines *AB* and *CD* and a transversal *BE* are angles…
b) Consecutive angles for *AB* and *CD* and a transversal *BD* are angles…
c) Alternate angles for lines *AB* and *CD* and a transversal *BE* are angles…
d) Alternate angles for *AB* and *CD* and a transversal *BD* are angles…
e) Corresponding angles for lines *AB* and *CD* and a transversal *BE* are angles…
f) Corresponding angles for *AB* and CD and a transversal *BD* are angles…

5. Two lines *a* and *b* intersect line *c* forming 8 angles. How many of them may be obtuse? How many of them may be right? Explain your answer.

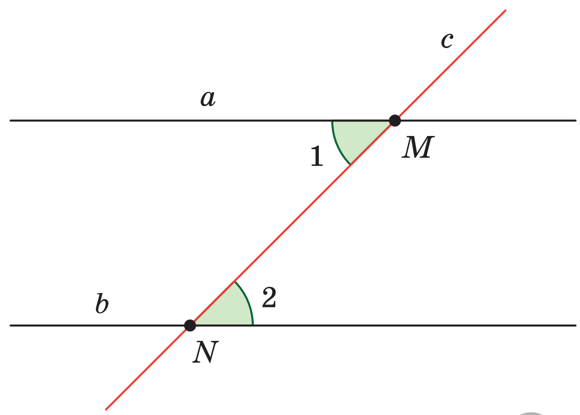
3.2 Conditions of parallel lines

Since we work a lot with lines in geometry, we should be able to find parallel lines among them. You already know the axiom of parallel lines and you know the types of angles formed by two lines and a transversal. Now we are going to apply this knowledge in order to find conditions that make it evident if lines are parallel or not.

These conditions are called conditions of parallel lines.

1. Comment the proof of the theorem.

If a transversal intersects two lines so that alternate interior angles are congruent, then the lines are parallel.



Given: a, b — lines, c — transversal,

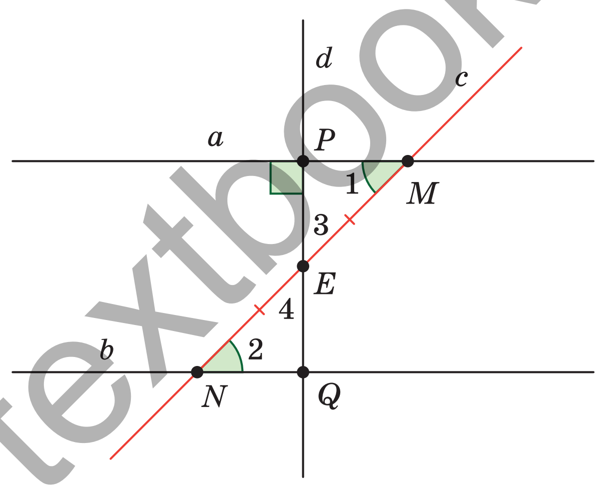
$a \cap c = M$,

$b \cap c = N$,

$\angle 1 = \angle 2$.

Prove: $a \parallel b$.

Proof:



Asssume that the line c intersects the lines a and b at the points M and N, respectively, and $\angle 1 = \angle 2$.

Find the midpoint E of the segment MN.

Draw a line d through the point E which is perpendicular to the line a Mark the intersection points of the line d with the lines a and b and label them P and Q respectively.

$\angle 1 = \angle 2$ (as given),
$EM = EN$ (by construction),
$\angle 3 = \angle 4$ (as vertically opposite), $\Rightarrow$

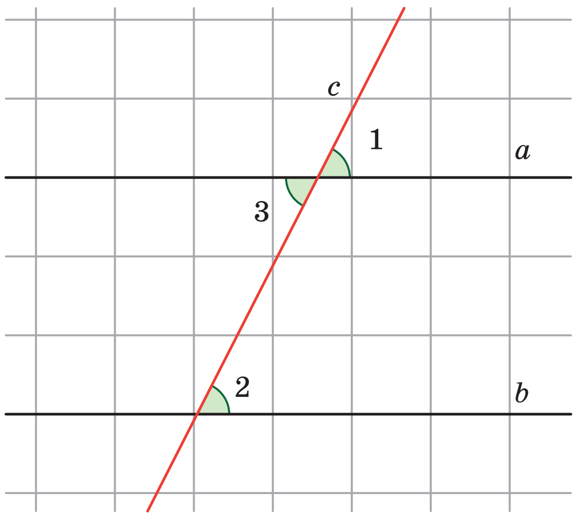
$\Rightarrow \triangle MPE = \triangle NQE$ (by the 2nd condition of congruence of triangle).

That means, $\angle MPE = \angle NQE = 90°$, and the lines a and b are perpendicular to the same line d, which means they are parallel.

Which proves the theorem.

2. Ruslan wrote a proof for other two conditions of parallel lines but some writing was wiped out. Help Ruslan to restore the original writing:

if a transversal intersects two lines so that alternate interior angles are equal, then the lines are parallel.

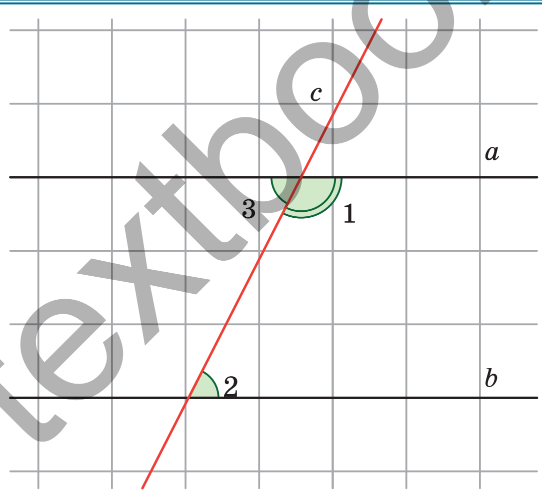


Given: a, b — lines, c — transversal.
$\angle 1 = \angle 2$.

Prove: $a \square b$.

Proof:
$\angle 3 ... \angle 1$ $\angle 3$ ... $\angle 1$ (why?)
$\angle$ ... and $\angle 2$ alternate, hence $\angle 2 = \angle ...$, that means a ... b (why?)
Which proves the theorem.

If consecutive angles formed by two lines and a transversal add up 180°, then the lines are parallel.



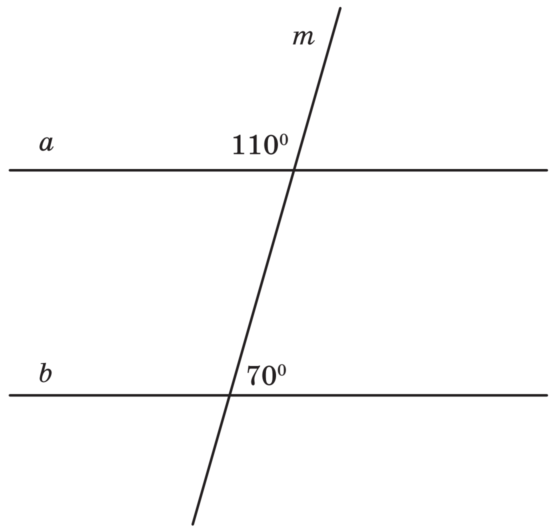
Given: a, b — lines, c — transversal.
$\angle 1 + \angle 2 = 180°$.

Prove: $a \square b$.

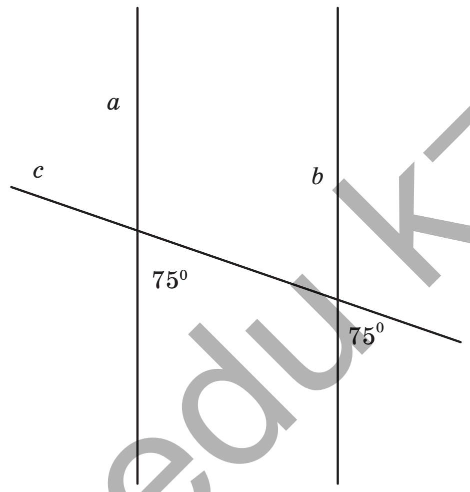
Proof:
$\angle 3 + \angle ... = 180°$ (since the angles are supplementary adjacent).
$\angle 1 + \angle 2 = 180°$ (as given, $\angle 2 = \angle 3$ (why?), hence $a \square b$ (why?).
Which proves the theorem.

3. Solve problems using given drawings.

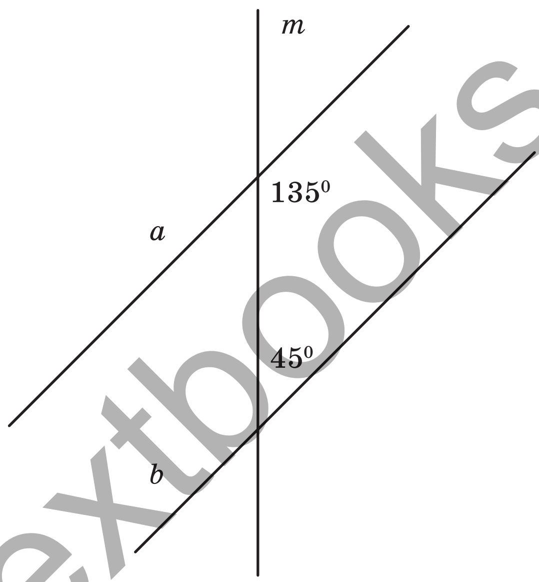
Given: a, b — lines, m — transversal.
Prove: $a \parallel b$.



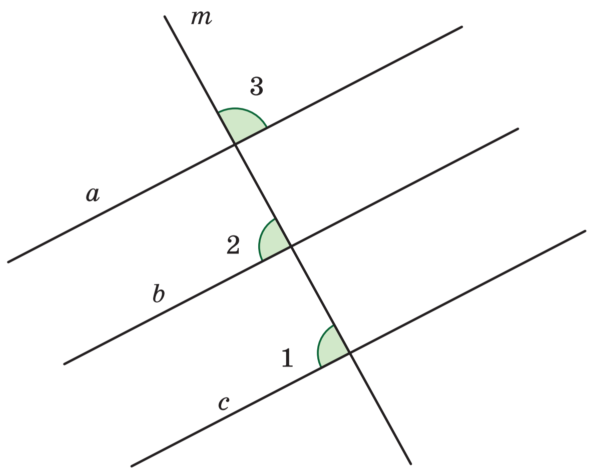
Given: a, b — lines, c — transversal.
Prove: $a \parallel b$.



Given: a, b — lines, m — transversal.
Prove: $a \parallel b$.



Given: a, b, c — lines, m — transversal,
$\angle 1 = \angle 2$,
$\angle 2 + \angle 3 = 180°$.
Prove: $a \parallel c$.



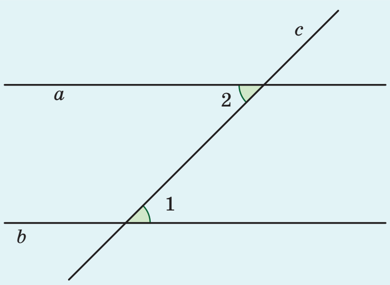
3.3 Properties of parallel lines. Problem solving

Consider the application of conditions of parallel lines in problem solving.

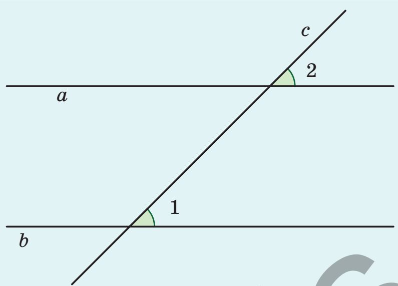
1. Formulate conditions of parallel lines.

REMEMBER!

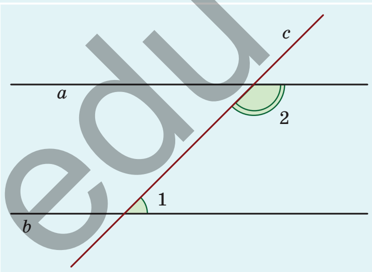
Conditions of parallel lines.



if a transversal intersects two lines so that alternate angles are equal, then the lines are parallel.



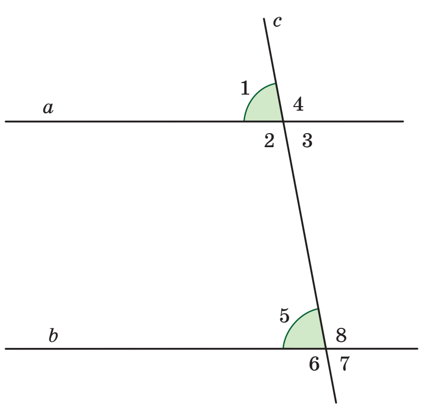
if a transversal intersects two lines so that corresponding angles are equal, then the lines are parallel.



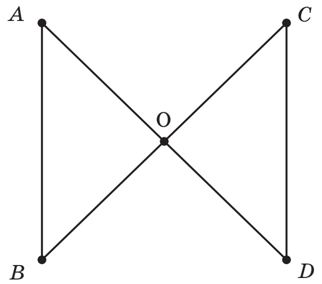
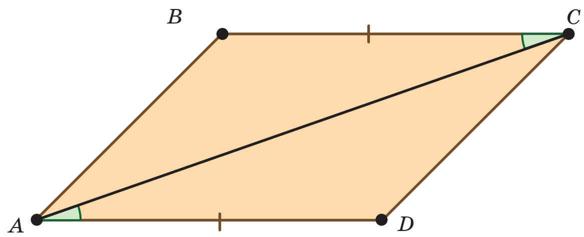
If consecutive angles formed by two lines and a transversal add up 180°, then the lines are parallel.

2. Damir drew lines a and b and a transversal c as shown in the drawing. Angle 1 equals angle 5. Are the following correct:

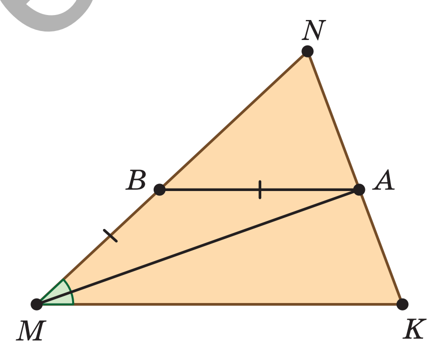
a) $\angle 4=\angle 6$;	b) $\angle 2=\angle 6$;	c) $\angle 2=\angle 7$;
d) $\angle 3=\angle 8$;	e) $\angle 2=\angle 5$;	f) $\angle 3=\angle 7$;
g) $\angle 2=\angle 8$;	h) $\angle 1=\angle 6$;	i) $\angle 3=\angle 5$;
j) $\angle 1=\angle 7$;	k) $\angle 2+\angle 5=180°$;	l) $\angle 3+\angle 8=180°$;
m) $\angle 1+\angle 6=180^c$;	n) $\angle 5+\angle 7=180°$;	o) $\angle 3+\angle 2=180°$;
p) $\angle 3+\angle 5=180°$;	q) $\angle 4+\angle 7=180°$;	r) $\angle 3+\angle 7=180°$.



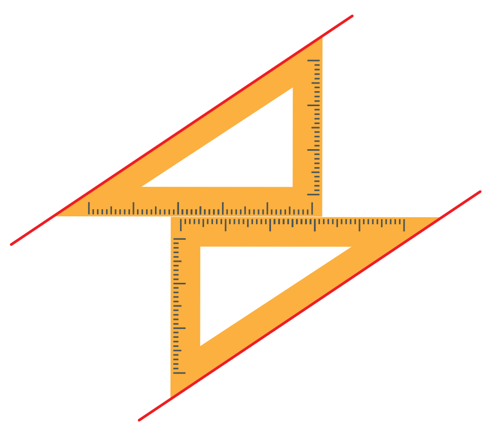
3. Solve the problems using given drawings.

	**Given:** $AD \cap CB = O$, $AO = OD$, $BO = OC$. **Prove:** $AB \parallel CD$.
	**Given:** $BC = AD$, $\angle BCA = \angle CAD$. **Prove:** $BC \parallel AD$, $AB \parallel CD$.

4. Given triangle MNK, the bisector of the angle M intersects the side NK at point A. Point B is marked on the side MN, so that $MB = AB$. Prove that lines BA and MK are parallel.



5. Arman says he can use two identical set squares to draw parallel lines. He presented his solution on the board. Is it correct? Why? Explain your answer.



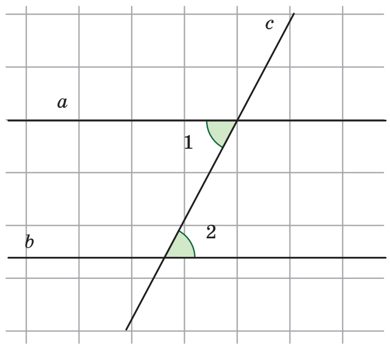
6. Given rectangle $ABCD$ $\angle A = 52^\circ$, $\angle B = 128^\circ$, $\angle C = 56^\circ$. Are sides AB and CD parallel? Are AD and BC parallel? Explain why.

3.4 Properties of parallel lines

Building up from your knowledge of the conditions of parallel lines and axiom of parallel lines we are going to learn about properties of parallel lines.

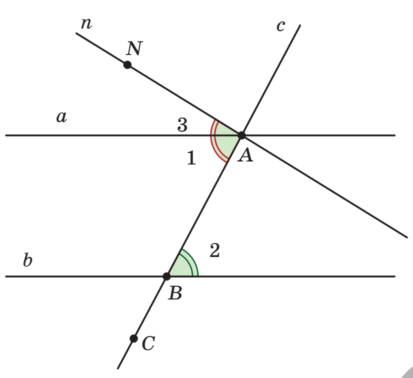
1. Comment the poof of a property of parallel lines.

If a transversal intersects two parallel lines, the alternate angles are equal.



Given: $a \parallel b$, c — transversal.
Prove: $\angle 1 = \angle 2$.

Proof: Assume that $\angle 1 \neq \angle 2$

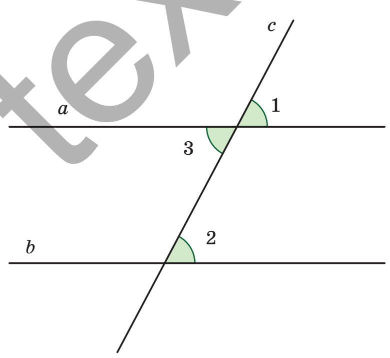


Draw a line n through point A so that $\angle 2 = \angle 3$. Then the angles 2 and 3 are alternate for lines n and b and a secant c. That means n and b are parallel (why?) That means that you can draw two lines that are parallel to line b through the point A, which is impossible (why?). That means the assumption is incorrect and $\angle 1 = \angle 2$.

Which proves the theorem.

2. Look through and comment the proof plan for the property of parallel lines. Create your own proof plan for the second statement.

If a transversal intersects two parallel lines, the corresponding angles are equal.

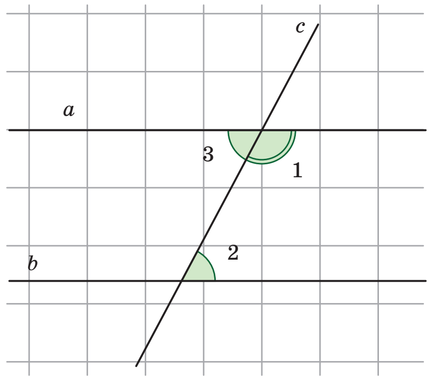


Given: $a \parallel b$,
c — transversal.
Prove: $\angle 1 = \angle 2$.

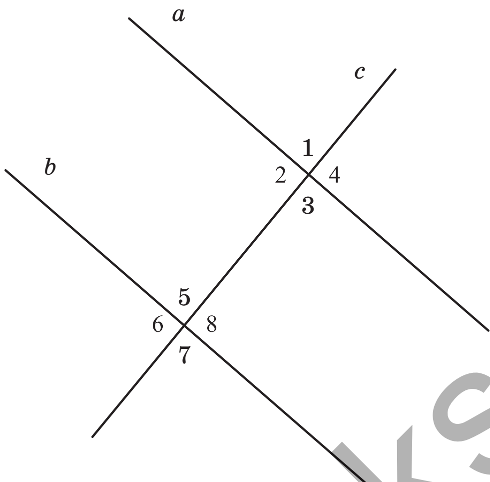
Proof

Statement	Argumentation
1. $a \parallel b$	As given
2. $\angle 3 = \angle 2$	As alternate angles formed by two parallel lines a and b and a transversal c
3. $\angle 1 = \angle 3$	As vertically opposite angles
4. $\angle 2 = \angle 1$	As conditions 2 and 3 are satisfied

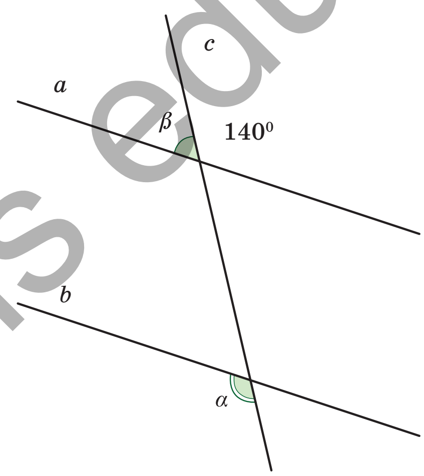
If a transversal intersects two parallel lines, the sum of consecutive angles equals to $180°$.

		Proof	
		Statement	**Argumentation**
	**Given:** $a \parallel b$, c — transversal. **Prove:** $\angle 1 + \angle 2 = 180°$.	1. 2. …	

3. Solve the problems using given drawings. Explain your answer.

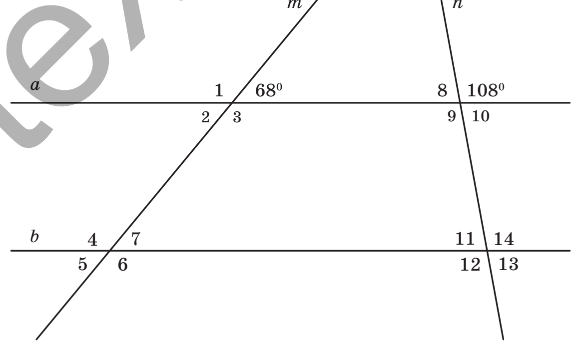


Given: $a \parallel b$,
c — transversal, $\angle 2 = 110°$.
Find: All unknown angles.

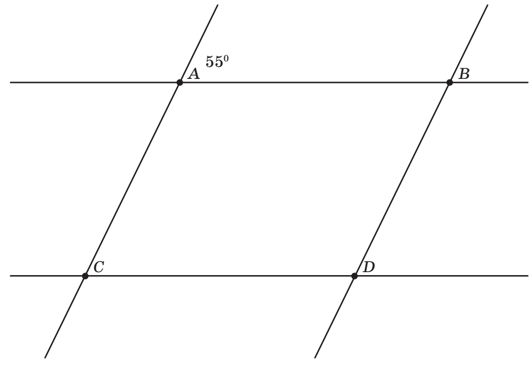


Given: $a \parallel b$,
c — transversal.
Find: $\angle \alpha$, $\angle \beta$.

Given: $a \parallel b$, m, n — секущие.
Find: All unknown angles.



Given: $AB \parallel CD$, $AC \parallel BD$.
Find: $\angle ABD$, $\angle BDC$, ACD, BAC .



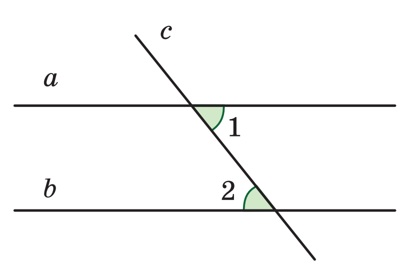
4. Intersection of two parallel lines and a transversal forms eight angles. Degree measures of two angles have a ratio of 1.5: 3. Find the degree measures of the remaining angles.

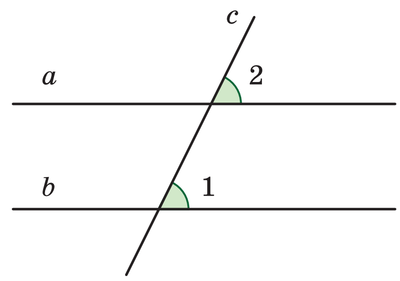
5. The difference of two interior consecutive angles formed by two parallel lines and transversal, equals to 50˚. Find the degree measures of these angles.

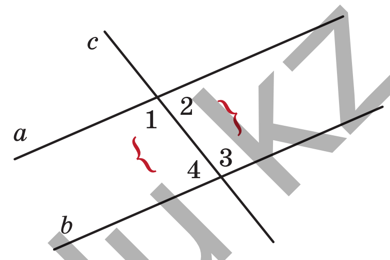
3.5 Properties of parallel lines. Problem solving

Consider the application of properties of parallel lines in problem solving.

1. Formulate the properties of parallel lines.

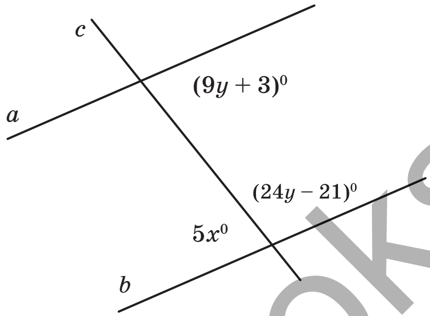




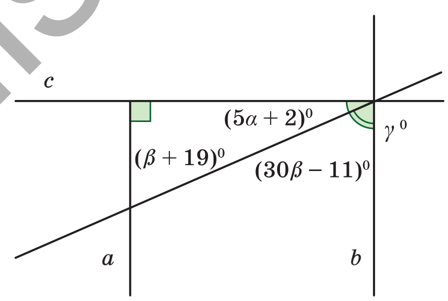


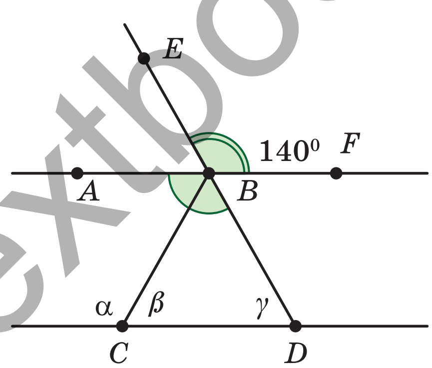
2. Solve the problem using given drawings.

Given: $a \parallel b$.
Find: x, y.

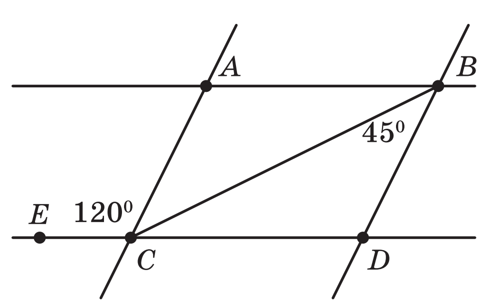


Given: $a \parallel b$.
Find: α, β, γ.

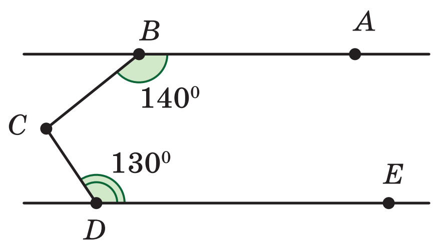




Given:
$AB \parallel CD$, BC — **bisector** $\angle ABD$.
Find: α, β, γ.

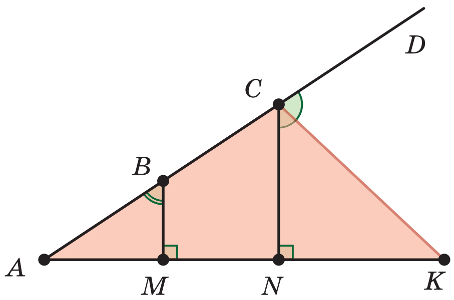


Given:
$AB \parallel CD, AC \parallel BD$.
Find: Angles of the triangle ACB



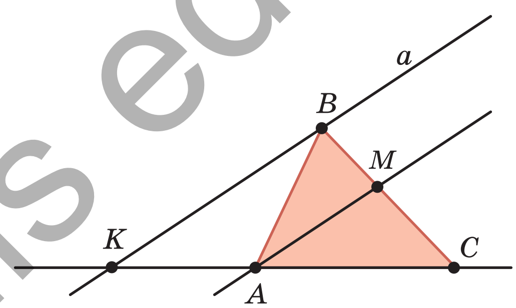
Given:
$AB \parallel DE$,
$\angle ABC = 140°$,
$\angle CDE = 130°$.

Prove: $BC \perp CD$.



Given:
$BM \perp AK$,
$CN \perp AK$,
CK — bisector $\angle DCN$, $\angle ABM = 44°$.
Find: $\angle ACK$.

3. Solve the problems:



a) Arman drew a line through the vertex B of the triangle ABC. This line is parallel to the bisector AM of angle A. The line intersects AC at the point K. What is the type of the triangle BAK?

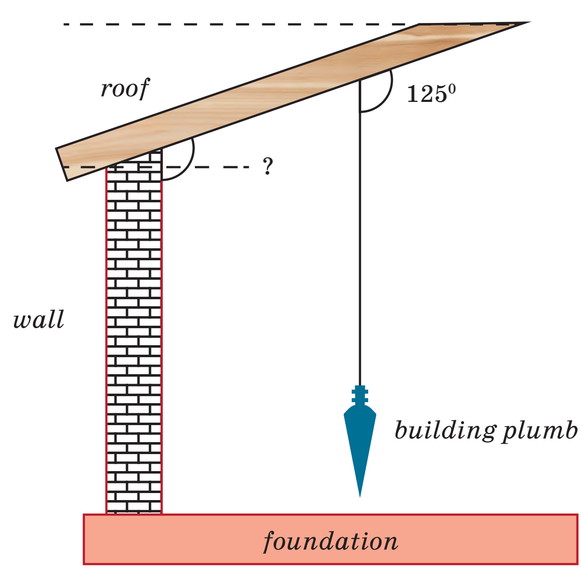
b) Arman drew a line that passes through the vertex C of the triangle ABC. This line is parallel to the bisector BD and intersects the continuation of the side AB at the point M. Find angles of the triangle MBC, if $\angle ABC = 110°$?

c) Arman states that bisectors of interior alternate angles formed by two parallel lines and a transversal, intersect. Is Arman right?

4. When a line intersects two parallel lines it forms eight angles. Seven angles are add up to. Find the size of the angles.

5. Given two parallel lines and a segment which endpoints belong to the lines. *M* is the midpoint of the segment. Damir drew a line through the point *M* that intersects the parallel lines at the points *A* and *B*.

6. Construction workers use a building plumb to check the evenness of the wall. This device is made of a thin thread and a load on the end. The gravity pulls the load down straining the thread which takes a constant position, perpendicular to the ground. The angle between the plumb and the roof was 125⁰. What angle should the wall and roof of the house have so that the wall is perpendicular to the foundation of the house? Why? Explain your answer.



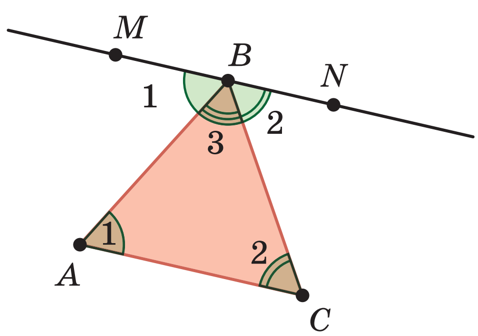
3.6 Sum of angles of triangle

The application of properties and conditions of parallel lines allows studying a large number of properties of various geometric figures. Look carefully at the illustrating example below.

REMEMBER!

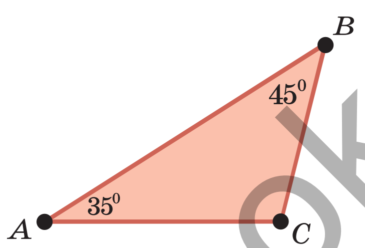
Angles of triangle add up to 180^0.

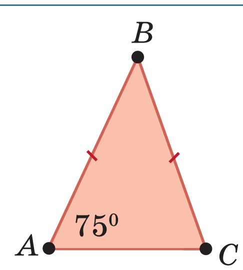
1. Marat made a formula of sum of angles of triangle. Is his argumentation correct? Comment on his solution.

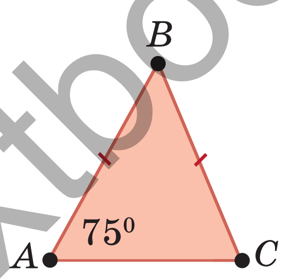


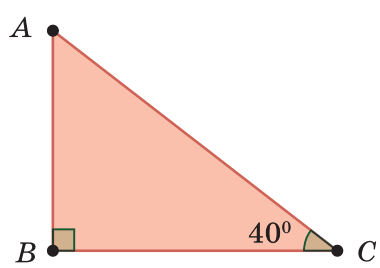
Given: ABC — triangle.
Find: $\angle A + \angle B + \angle C$.
Solution::
Draw a line $MN \parallel AC$.
$\angle A = \angle MBA$, $\angle C = \angle CBN$ (why?),
$\angle MBA + \angle ABC + \angle CBN = 180°$ (why?).
Hence, $\angle A + \angle B + \angle C = 180°$.
Answer: Angles of the triangle add up to $180°$.

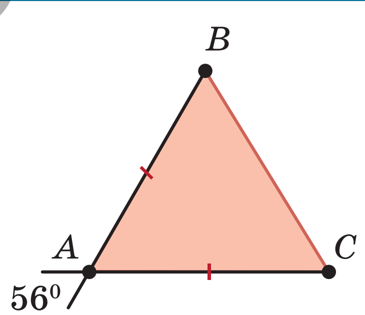
2. Solve the problems using ready made drawings. Find degree measure of all triangles.

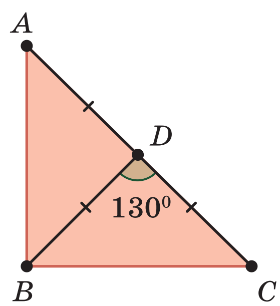




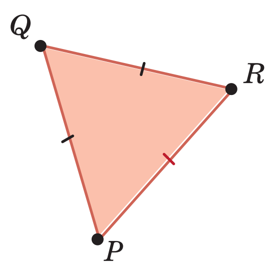
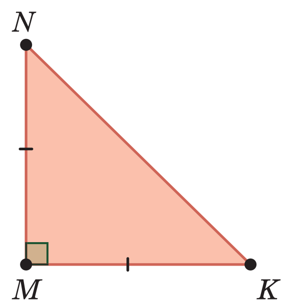
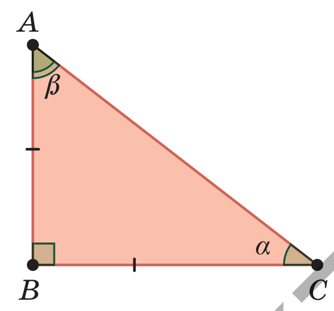








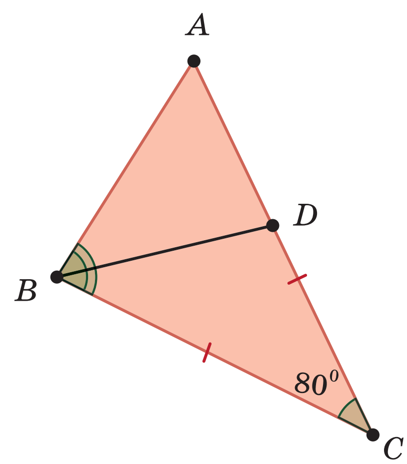
3. Fill in the gaps.

		
Angles of equilateral triangle equal to…	Angles of isosceles triangle equal to…	Acute angles of right triangle add up to…

4. Solve the problems.

a) One of the angles of a triangle is 1/4 of the second angle and 30˚ less than the third. What is the degree measure of the angles of the triangle?
b) Angles of a triangle relate to one other as 2:3:4. What is the degree measure of the angles of the triangle?

5. Solve the problems.



a) One of the angles of an isosceles triangle is 100˚. What is the degree measure of other angles of the triangle?
b) What is the degree measure of angles of an isosceles triangle if one of them is 4 times the other?
c) An angle at the vertex of an isosceles triangle *ABC* is 80˚. What is the degree measure of an angle formed by a bisector of the angle adjacent to the base and opposite lateral side of the triangle?
d) The angle formed by bisectors of the angles adjacent to the base of an isosceles triangle *PQR* is 126˚. What is the degree measure of angles of a triangle *PQR*.

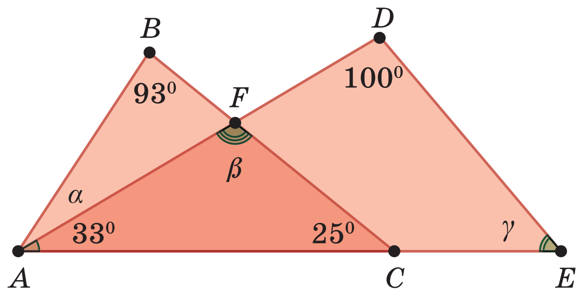
6. Given triangle *MNP* with an angle *M* of 40˚, an angle *N* of 80˚. Point *O* — point of intersection of angle bisectors *M* and *N*. What is the degree measure *NOP*?

7. A line parallel to the *PR* passes the vertex *Q* of the triangle *PQR*. This forms three angles with a vertex *Q*. Their degree measures relate as 4: 9: 5. Find the angles of the triangle and determine its type.

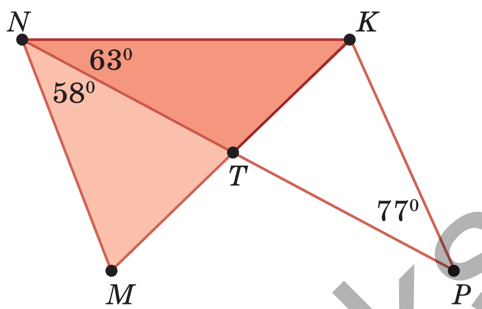
8. Arman has found a method of measuring an angle at the base of an isosceles triangle with 80˚ angle at the vertex. Then he halved the angle receiving two 40˚ angles. He subtracted 40˚ form 90˚ angle and received 50˚ angle. What is new about this method of finding angles? Explain your answer.

3.7 Sum of angles of a triangle. Problem solving

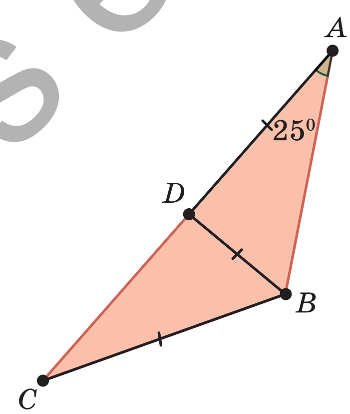
1. Solve the problems, using ready made drawings. Find unknown angles of a triangle.



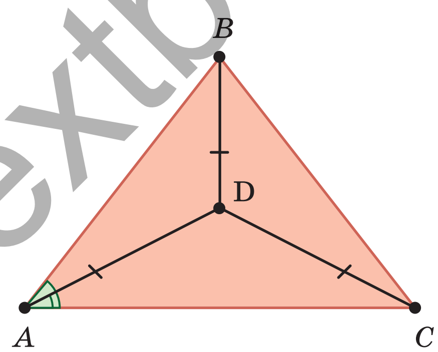
Find: α, β, γ.



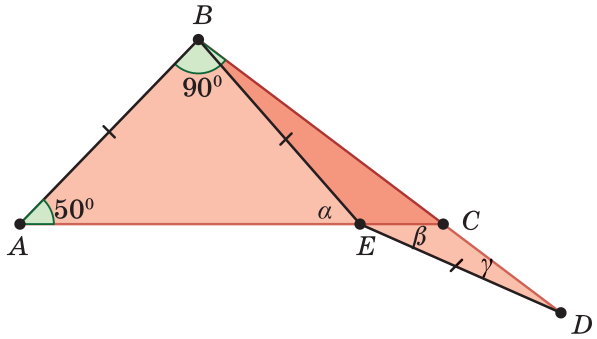
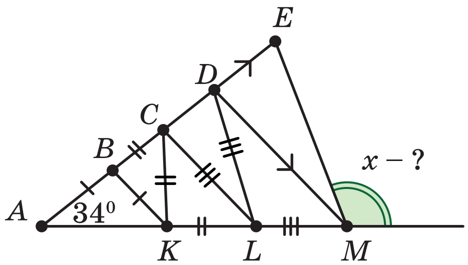
Given: KM — bisector of an angle K.
Find: Angles of MNT.



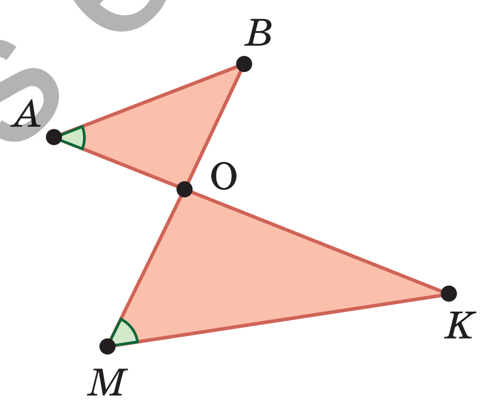
Given: ABC — triangle.
Find: Angles of a triangle ABC.



Given: ABC — triangle.
Find: Angles of a triangle ABC.

	**Given:** $\angle ABC = 110°$. **Find:** α, β, γ.
	**Find:** x.

3. Solve the problems:



a) Leila drew segments AK and BM, which intersect at point O. Angles M and A are equal. Is it true that angles B and K are also equal? Why? Explain your answer
b) Leila assumed that if two angles of one triangle are respectively equal to two angles of another triangle, then the third angles of the triangles are also equal. Is Leila right? Why? Explain your answer.

4. The angle bisector A of a triangle ABC divides it into two isosceles triangles. Find the angles of a triangle ABC.

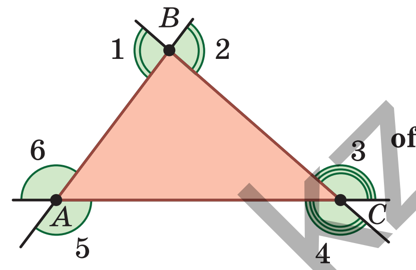
5. Solve the problems:
a) Given triangle ABC where $\angle A = 70^{\circ}$, $\angle C = 80^{\circ}$. Altitudes of the triangle drawn from the vertices A and C intersect at point M. What is the degree measure of $\angle AMC$?
b) A median DK is drawn in the triangle DEF. Find the angles of the triangle DEF if $\angle KDE = 70^{\circ}$, $\angle DKF = 140^{\circ}$.

6. Marat draw a triangle where the degree measure of each angle is expressed by a prime number. Can you draw a triangle like this? Explain your answer.

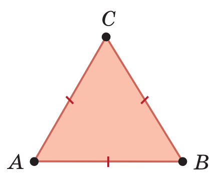
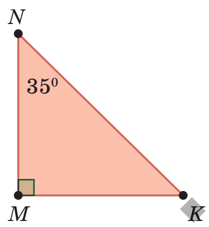
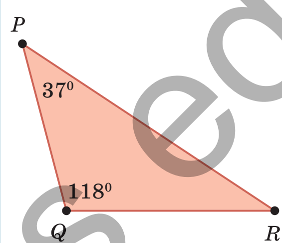
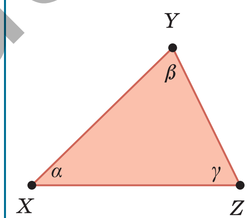
3.8 Exterior angle of a triangle

In this lesson we continue talking about angles of triangle and learn about another important geometry concept which is exterior angles of a triangle.

Exterior angles are angles adjacent to angles of a triangle.
$\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$ — exterior angles of triangle *ABC*.



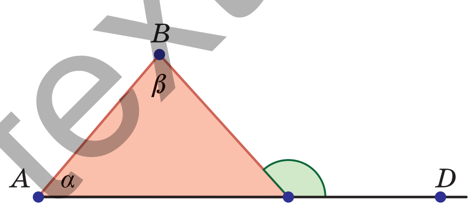
1. Find exterior angles of a triangle and fill out the table.

Triangle				
Exterior angles				
Sum of exterior angles of a triangle				

Is it correct that exterior angles add up to 360°?

2. Determine the type of a triangle if one of its interior angles equals to its adjacent angle.

3. Comment on the solution given below.
Given a triangle *ABC*. $\angle A = \alpha$, $\angle B = \beta$. Find exterior angle *BCD* of a triangle *ABC*.
Can you come up to your own solution of the problem?

	**Given:** *ABC* — triangle, $\angle A = \alpha, \angle B = \beta$. **Find:** $\angle BCD$.
Solution::	

ΔABC

↓

$\angle BAC + \angle ABC + \angle BCA = 180^\circ$

$\angle BCA$, $\angle BCD$ adjacent angles

↓

$\angle BCA + \angle BCD = 180^\circ$

↓

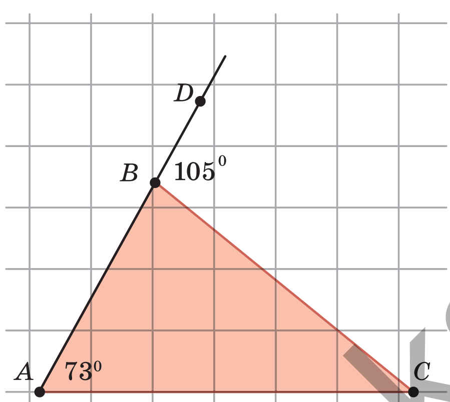
$\angle BAC + \angle ABC + \angle BCA = \angle BCA + \angle BCD$

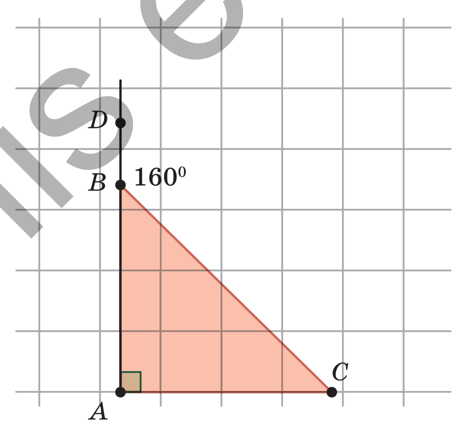
↓

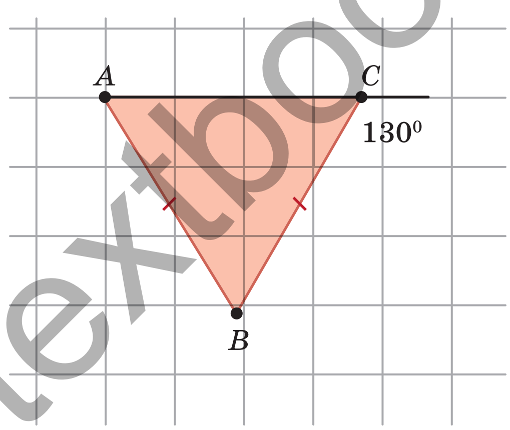
$\angle BAC + \angle ABC = \angle BCD$

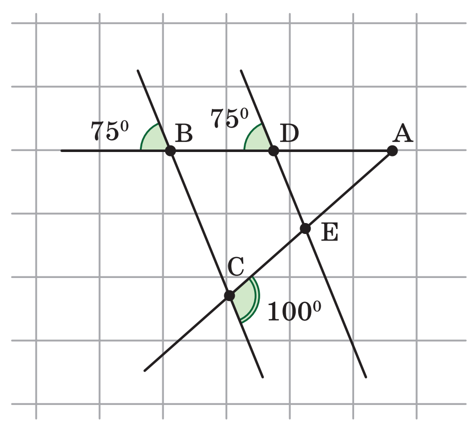
exterior angle of a triangle equals the sum of two angles not adjacent to it.

4. Solve the problems using ready made drawings. Find the unknown angles of the triangle *ABC*.









5. Solve the problems:

a) What is the ratio of the external angles of a triangle if the angles of the triangle are related as 2: 3: 4.

b) The degree measure of an exterior angle of a triangle is 64°. Degree measures of angles not adjacent to it are related as 3: 5. Find the angles of the triangle that are not adjacent to a given angle.

c) One of the exterior angles of the triangle is 154°. Find the angles of the triangle that are not adjacent to a given angle if the degree measure of one of them is 36° larger than the other.

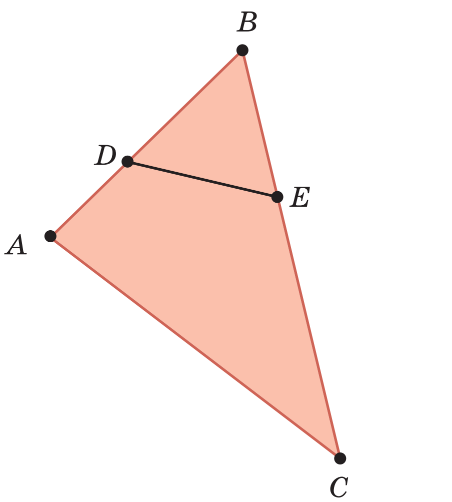
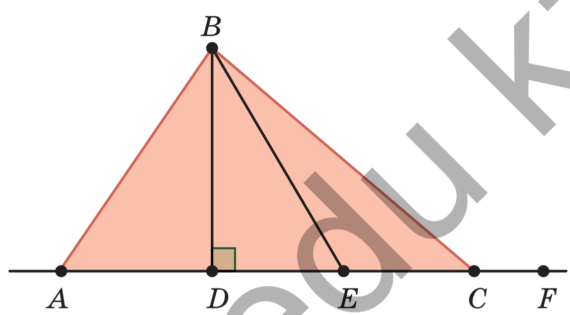
6. Find angles of an isosceles triangle if one of its exterior angles equals to:

a) 36°;
b) 146°.

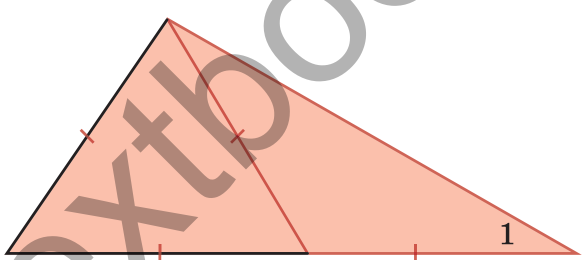
7. Determine the type of a triangle if one of its interior angles is 40° and one of its exterior angles is 110°.

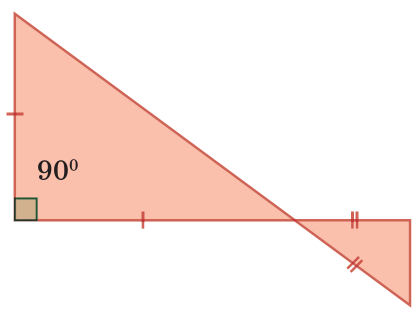
3.9 Exterior angle of a triangle. Problem solving

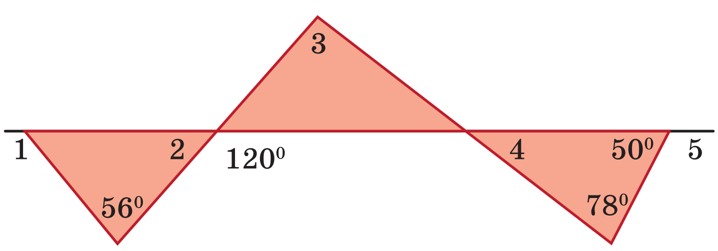
1. Examine the drawings and answer the questions. Explain your answer.

	
1. List all external angles of a triangle *BDE*. 2. May they be right, obtuse or acute?	1. List all external angles at the vertices *D* and *E*. 2. In relation to what angles an angle *BCF* is external? 3. Find common external angle of the triangles *ADB*, *ABE* and *ABC*.

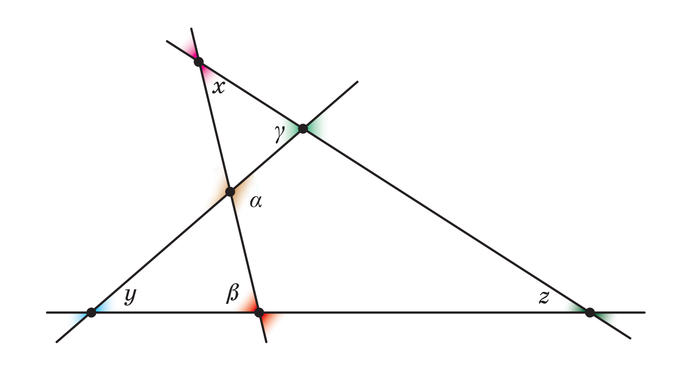
2. Find unknown angles of a triangle.

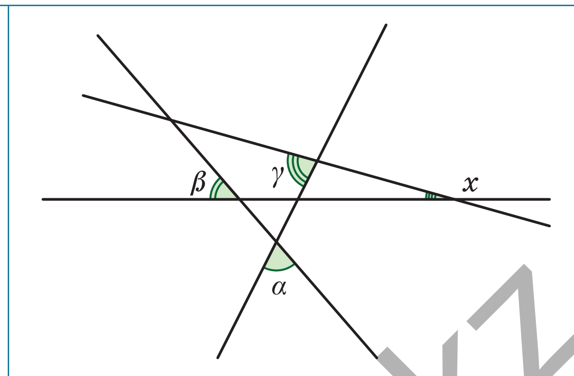






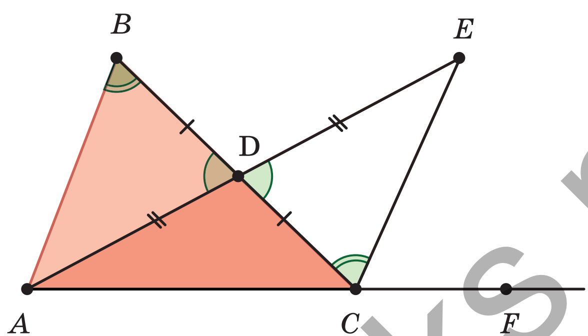
3. Find degree measures of angles x and y, if $\alpha = 80°$, $\beta = 60°$, $\gamma = 70°$.





4. Two angles of a triangle equal 10° and 70°. Find the angle between an altitude and a bisector drawn from the vertex of the third angle of a triangle.

5. Marat proved that exterior angle of a triangle is greater than every internal angle not adjacent to it. Comment his proof.



Given: $\triangle ABC$.
Prove: $\angle BCE < \angle BCF$.

Statement	Explanation
1. Draw a median AD of a triangle ABC and draw it beyond a point D.	
2. Mark a segment $AD = DE$	based on the axiom of making off a segment equal to a given one
$\angle ADB = \angle CDE$	vertically opposite angles
3. $BD = DC$	by construction
$\triangle ADB = \triangle DEC$	by the first condition of congruence of triangles
$\angle ABC = \angle BCE$	
$\angle BCE < \angle BCF$	as angle BCE is a part of an angle BCF

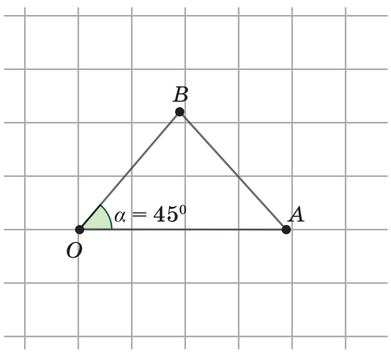
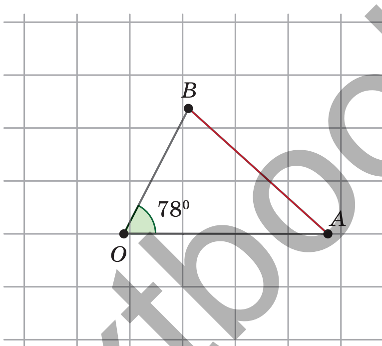
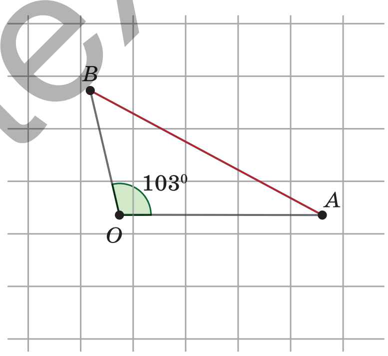
Which proves the theorem.

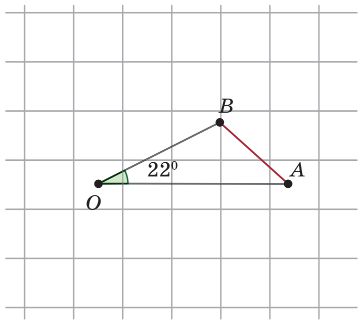
7. Points N and M are marked on the lateral sides AB and AC of a triangle ABC where $AN = NM = MB = BC$. Find angles of a triangle ABC.

3.10 The ratio between the sides and angles of a triangle

An important aspect of studying a triangle is the ratio between its sides and angles. Now we are going to explore this ratio and its applicability in problem solving.

1. Marat drew a 45° angle *AOB*. He marked points *A* and *B* on the sides of the angle and measured the length of the segment *AB*. Then he changed the degree measure of the angle *AOB*, by increasing or decreasing it. However, points *A* and *B* remained stationary. Each time, he measured the length of the segment *AB* he recorded the data in the table. Recover Marat's records. What conclusion did he make?

	Drawing	Degree measure of an angle *AOB*	The length of a side *OA*	The length of a side *OB*	The length of a side *AB*
1					
2					
3					

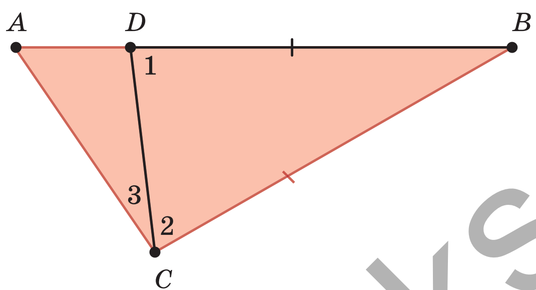
4					

Use a protractor to measure angles *OBA* and *OAB* and fill in the table. What dependence between sides and their opposite angles can you notice?

2. Marat conducted a research and proved a theorem:

> **There is the largest angle opposite to the largest side, and there is the largest side opposite to the largest angle.**

Comment on the proof of the first part of a theorem and suggest a plan.



Given:
$\triangle ABC$,
$AB > BC$.

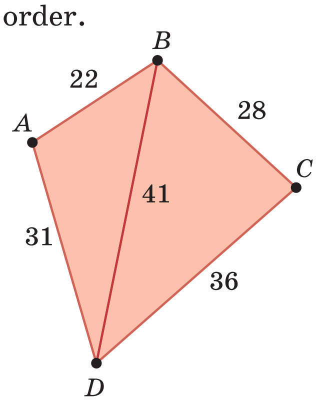
Prove: $\angle BCA > \angle BAC$.

Proof: from the point *B* mark off a segment $BD = BC$

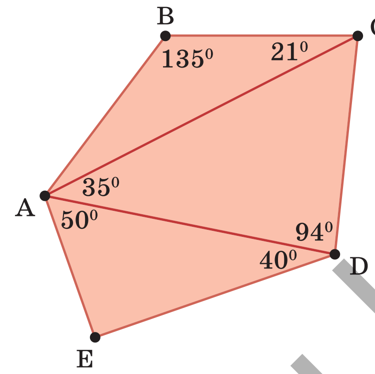
Statement	Argumentation
1. $BD = BC$	Based on the axiom of marking off a segment equal to a given one.
2. $\angle 1 = \angle 2$	Since a triangle ***BCD*** is isosceles
3. $\angle BAC < \angle 1$	Since the exterior angle of a triangle is larger than the angle not adjacent to it.
4. $\angle 2 + \angle 3 = \angle BCA$	As sum of two angles
5. $\angle 2 < \angle BCA$	Since $\angle 2$ is a part of $\angle BCA$
6. $\angle 1 < \angle BCA$	Since $\angle 1 = \angle 2$
7. $\angle BCA > \angle BAC$	Based on 3 and 6.

3. Use the theorem of ratio between sides and angles to:

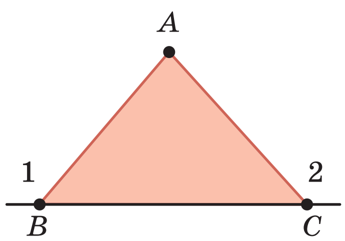
Put angles of triangles *ABD* and *BDC* in ascending order.



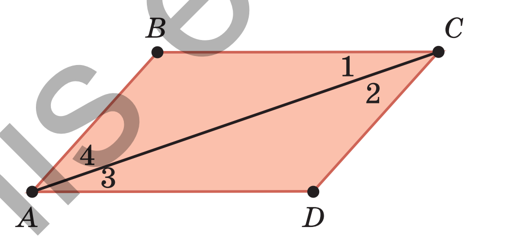
Put the sides in descending order.



4. Use ready made drawings to prove that:



if $AB>AC$, then $\angle 1>\angle 2$



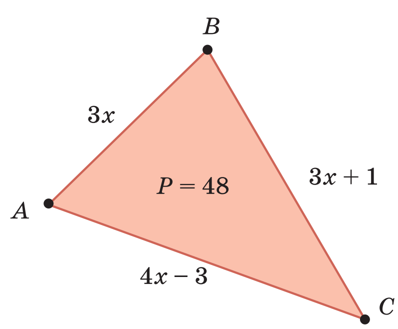
if $\angle 1=\angle 3$ and $\angle 3<\angle 2$, then $AC>AD$

5. Given triangle *ABC*. Put the angles in ascending order if:

a) $AB = 11$ cm; $BC = 8$ cm; $AC = 9$ cm;

b) $AB = 30$ cm; $BC = 18$ cm; $P_{ABC} = 60$ cm;

c)

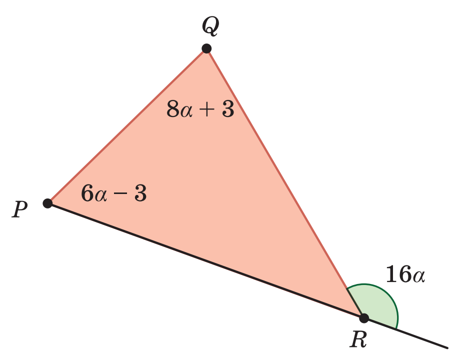


6. Compare sides of a triangle *PQR*, if:

a) $\angle P=38°, \angle Q=67°$;

b) $\angle P=132°, \angle R=2$;

c)



3.11 Triangle inequality

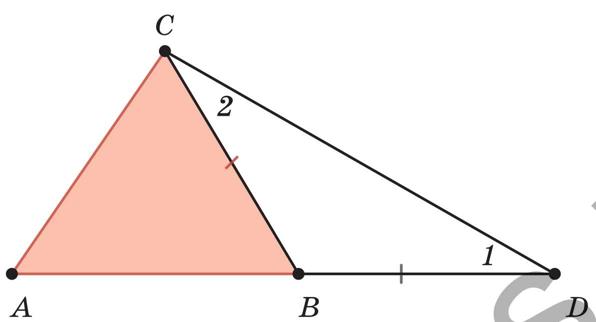
In everyday life we often have to solve the problem of calculating the optimal route, finding the optimal size. This where knowledge about the ratio of angles and sides of a triangle and the triangle inequality become handy.

1. Conduct a research.

1. Cut out paper strips of 7, 12 and 9 cm. Arrange the stripes into a triangle.
2. Do the same thing with strips 7 cm, 14 cm, 7 cm and 5 cm, 16 cm, 7 cm long. Is it always possible to do this? Compare the sum of the lengths of any two sides of the triangle with the third side. What do you notice?

2. Comment the proof of the theorem below. Add any necessary explanations to it.

Sum of two sides of triangle greater than third side.



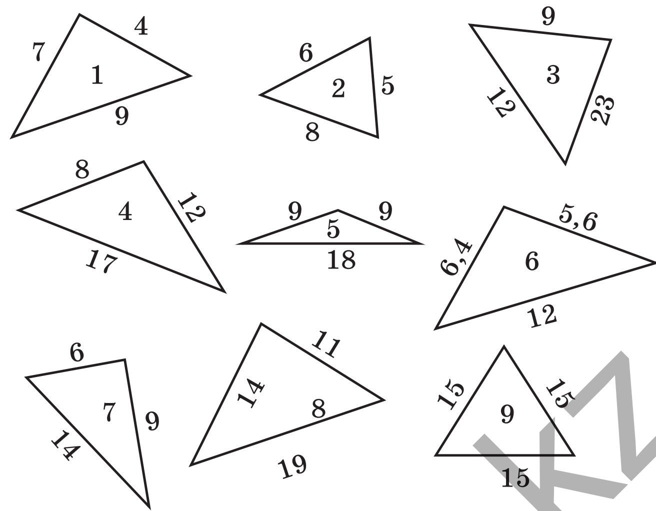
Given: ABC — triangle.
Prove: $AC < AB+BC$.
Proof: from a point B draw a segment $BD = CB$.

Statement:	Explanation:
1. $CB = BD$	
2. $\angle 1 = \angle 2$	
3. $\angle 2 < \angle ACD$	
4. $CDA < ACD$	
5. $AC < AD$	
6. $AD = AB + BD$	
7. $AB + BC > AC$	

4. Does a triangle with the following dimensions exist:

a) its sides equals 15 cm, 4 cm и 8 cm; 7 cm, 12 cm, 8 cm; 1 cm, 2 cm, 3 cm?
b) its sides relate to one other as 1:2:3; 2:2:4; 2:3:6?

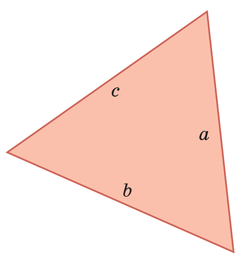
3. Meruert drew triangles and said that they all exist. Is she right?



5. Solve the problems:

a) Given an isosceles triangle with sides of 24 and 9 cm long. What side is the base of the triangle?
b) How many different triangles are there with the sides represented by integer numbers in centimetres and the perimeter of 12 cm?

6. Given two



a	b	c
8	12	7
5	2	16

7. Temple builders of Ancient Egypt used a rope with 13 equidistant knots. They used it to make triangles, where knots represented their vertices.

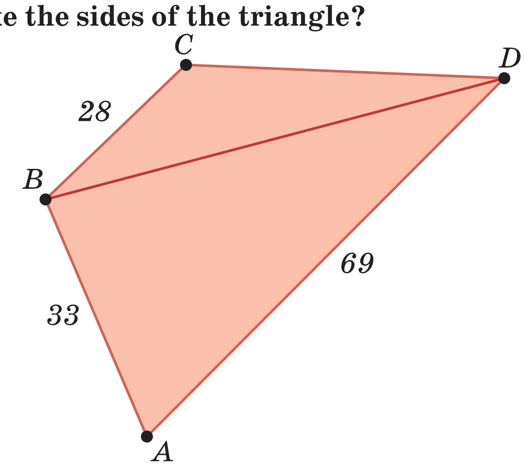
How many triangles can you make using the rope? Illustrate your solution.

8. Is the following correct:

1) each side of the triangle is larger than the difference of two other sides;
2) if $AC + CB = AB$, then points A, B and C lay on the same line?
Explain your answer.

9. Spikes *A*, *B*, *C* and *D* were driven on a plot of land as shown in the drawing. To what extent can a side *CD* vary within a given plot?

10. Given sides 5; 3*a*; 5*a*. What integer values of a make the sides of the triangle?

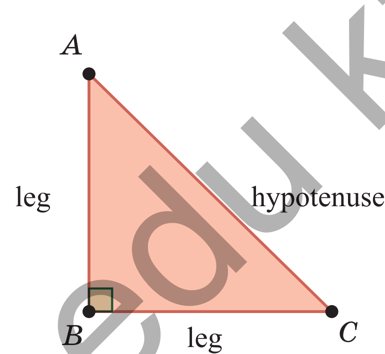


3.12 Right triangle. Conditions of congruence of right triangles

You have already learned about a triangle, you know its types and can recognize congruent triangles. On this lesson we are going to learn more about a right triangle.

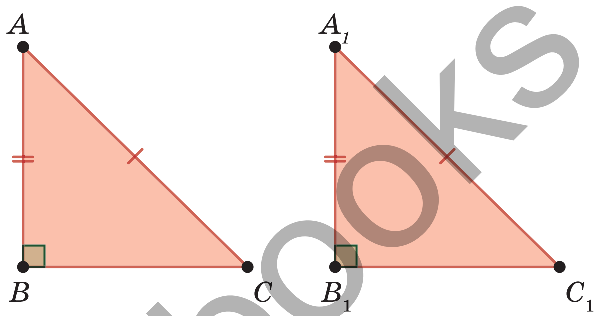
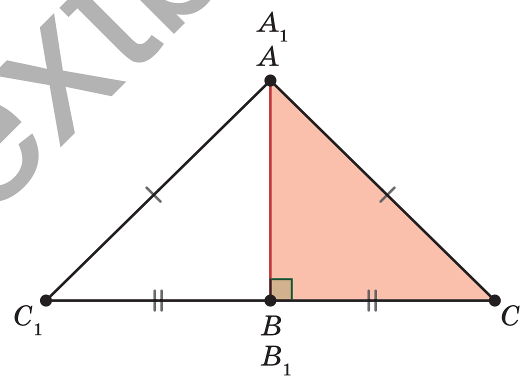
> A triangle where one of its angles is right angle and other two angles add up to 90°. The side of a right triangle opposite to a right angle is called the hypotenuse, and the other two sides are called legs.

Right triangle

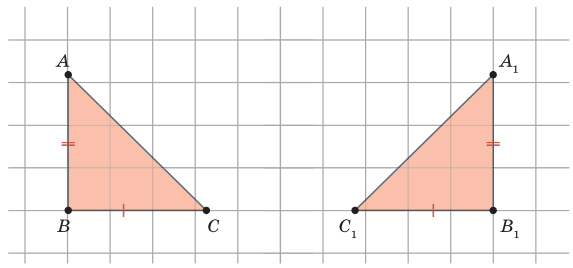
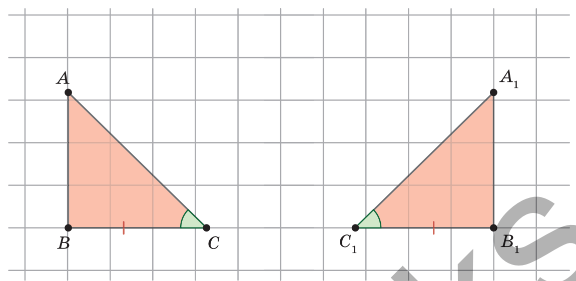
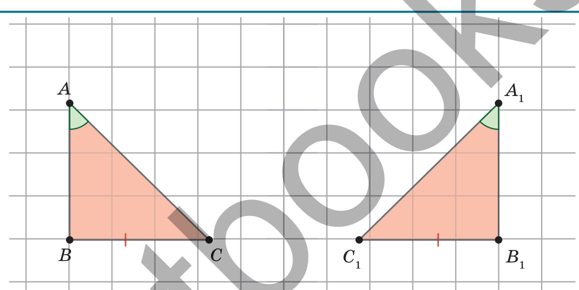
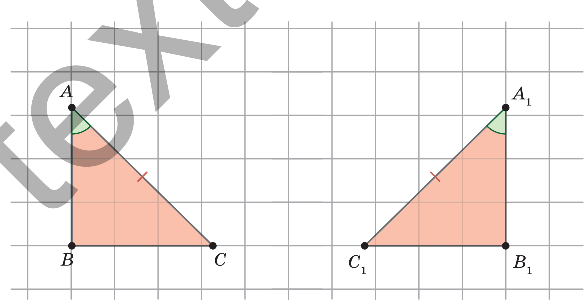


All conditions of congruence of triangles are applicable to right triangles.

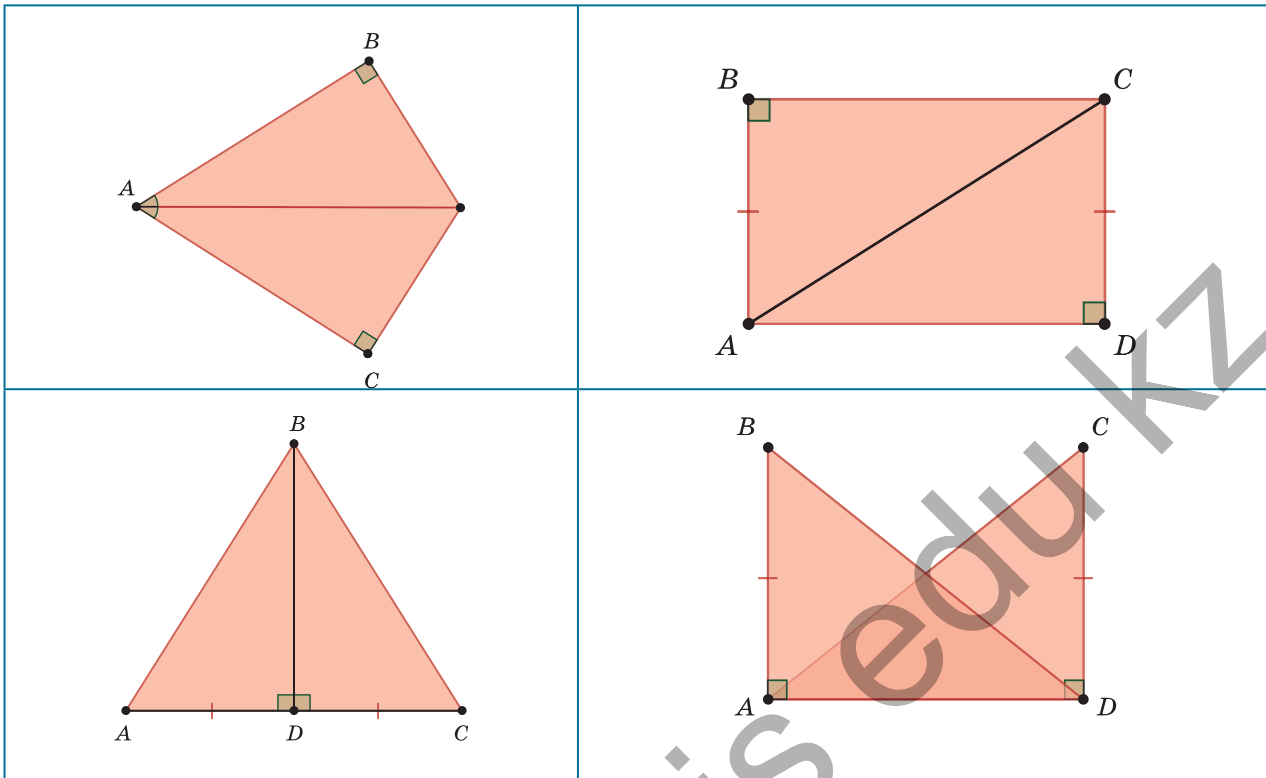
1. Refer to the drawings to formulate the conditions of congruence of right triangles and prove them.

Condition	Description
	(by a leg and a hypotenuse) If a hypotenuse and a leg of one right triangle are respectively equal to a hypotenuse and a leg of another right triangle, then these triangles are congruent.
	**Given:** $\triangle ABC$ $(\angle B = 90°)$, $\triangle A_1B_1C_1$ $(\angle B_1 = 90°)$, $AC = A_1C_1$, $AB = A_1B_1$. **Prove:** $\triangle ABC = \triangle A_1B_1C_1$.

Proof:
Superimpose the triangles ABC and $A_1B_1C_1$ так, so that a vertex A superimposed with a vertex A_1, a vertex B with a vertex B_1, and points C and C_I lay on different sides of line AB. Since $\angle ABC + \angle A_1B_1C_1 = 90° + 90° = 180°$, then $\angle CB_1C_1$ is $180°$ and points C, B_I, N_1 are collinear. A triangle A_1C_1C isosceles ($A_1C = A_1C_1$), then A_1B_1 — is both a median and a bisector. Hence, $C_1B_1 = CB_1$. Therefore, triangles ABC and $A_1B_1C_1$ are congruent by the third condition of congruence of triangles.
Which proves the theorem.

Condition	Description
	(by two legs) If legs of one right triangle are respectively equal to the legs of another right triangle, then these triangles are congruent
	(by a leg and adjacent acute angle) If a leg and its adjacent acute angle of one right triangle are respectively equal to the leg and its adjacent acute angle of another right triangle, then these triangles are congruent
	(by a leg and an opposite acute angle) If a leg and the opposite angle of one right triangle are respectively equal to a leg and the opposite angle of another right triangle, then these triangles are congruent
	(by a hypotenuse and an acute angle) If a hypotenuse and an acute angle of one right triangle are respectively equal to a hypotenuse and an acute angle of another right-angled triangle, then these triangles are congruent

2. Use ready made drawings to prove that $\triangle ABD = \triangle ACD$.



3.13 Properties of right triangle

A right triangle has a number of properties that other triangles do not have. Their properties become very handy in problem solving. To prove these properties, we can use our previous knowledge about triangles (triangle inequality, ratio between angles and sides of a triangle).

1. Are the following statements correct? Explain why.

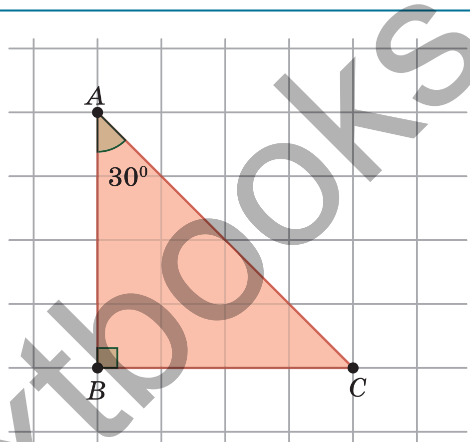
a) In a right triangle, the hypotenuse is always equal to a leg.
b) In a right triangle, the hypotenuse is always smaller than a leg.
c) In a right triangle, the hypotenuse is always larger than a leg.

2. Given a right triangle with sides 6 cm, 8 cm and 10 cm. What is the largest leg of the triangle? The smallest leg? How can you find it? Explain your answer.

3. Use a ruler, a set square and a protractor to draw right angle triangle *ABC* ($\angle A = 90^\circ$):
a) with a hypotenuse *BC* of 6 cm, and an angle *BCA* of 30°;
b) with a leg of 4 cm, and the angle of 60°formed by the hypotenuse and the leg. 60°

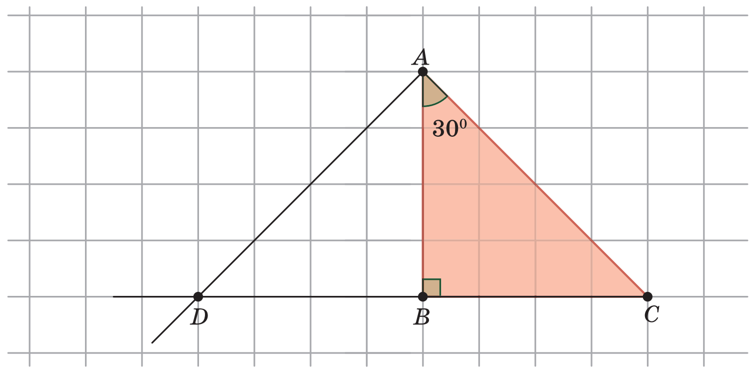
Compare the length of legs and compare them with the length of a hypotenuse?

4. Examine and comment the proof of a property of a right triangle. Add appropriate explanations to the proof.



Given: *ABC* — right triangle,
$BC = \frac{1}{2}AB$.

Prove: $\angle BAC = 30^\circ$.



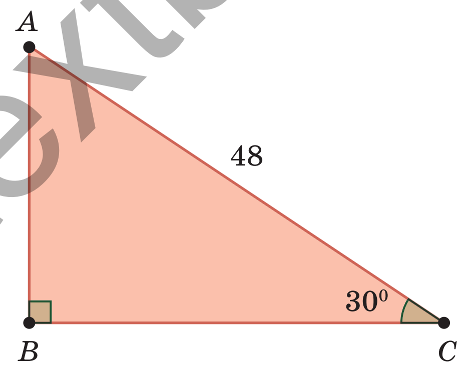
Proof:
Draw a segment $BD = BC$ beyond a point B.

In a right triangle the leg which is half the length of the hypotenuse is opposite the 30° angle.

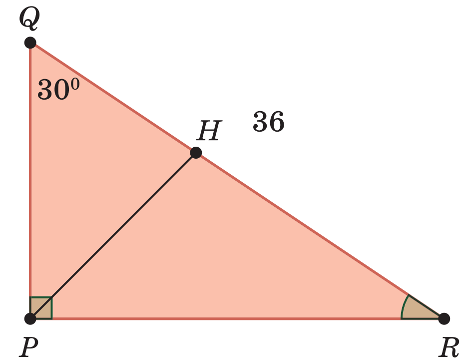
Statement	Argumentation
1. $BD = BC$	Based on the axiom of marking off a segment equal to the given one.
2. $\triangle ABD$ — isosceles	
3. $AC = CD = AD$	as $BC = \frac{1}{2}AB$ is given
4. $\angle A = 60^\circ$	Since $\triangle ACD$ is equilateral, and its sides are equal to 60°
5. $\angle BAC = \angle BAD = 30^\circ$	
Which proves the theorem.	

Is the opposite statement correct "In a right triangle the leg opposite the 30° angle is half the length of the hypotenuse"? Explain why.

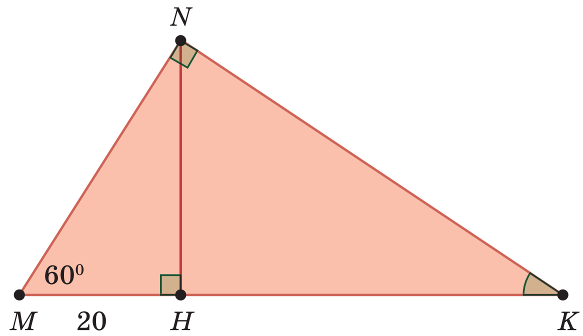
5. Solve the problems using ready made drawings. Find the unknown elements of the triangle.



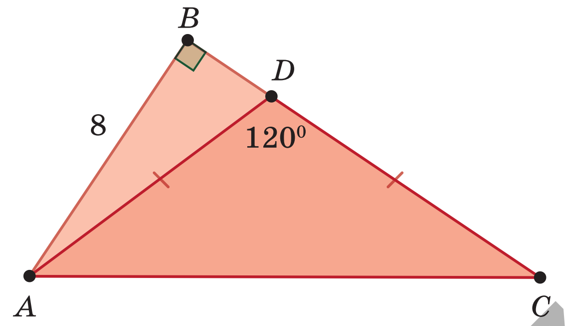
AB — ?



$PH \perp QR$ HR — ?



MN, HK — ?



AC — ?

6. Solve the problem:

Nurlan drew a right triangle ABC ($\angle C = 90^\circ$), with an angle A of 30°. Then he drew an altitude CH from the vertex of the right angle so that $BH = 9$. What is the length of its hypotenuse?

7. Use the Egyptian rope with 13 knots to make a right triangle. How many triangles did you get? What length are the sides of the triangle?

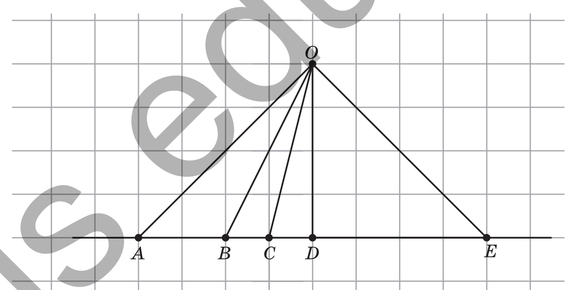
3.14 Perpendicular and slanted line

We often have to find the shortest distance from a given point to an object. What distance it will be? On this lesson we are going to look at this problem from a geometrical point of view.

We talked about a perpendicular to a line earlier this year. Today we are going to consider the geometric meaning of a perpendicular and its properties.

1. Akzhan needs to find the shortest distance from a point *O* to a line *a*. So she marked points *A*, *B*, *C* and *D* on the given line. Akzhan measured the lengths of the segments and examined the angles formed by the segments and this line. She recorded all the data into the table. Restore Akzhan's records. Which segment has the smallest length? Why? What assumptions do you have?

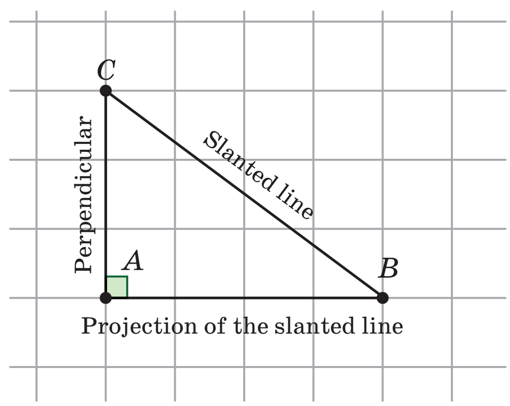
Segment	Angle	Length of segment
AO	$\angle OAD$	
BO	$\angle OBD$	
CO	$\angle OCD$	
DO	$\angle ODE$	
OE	$\angle OED$	



The shortest distance from a point to a line is the length of the perpendicular drawn from a given point to a given line.

Given the line *a* and the line *b* that passes the point *C* outside the line *a*. If *b* is perpendicular to *a*, then point *A* is a base of a perpendicular.

- **Segment *AC* is the perpendicular to *a*;**
- ***CB* is a slanted line from point *C* to *a*;**
- **Segment *AB* is a projection of the slanted line to *a*.**



2. Is the following correct:

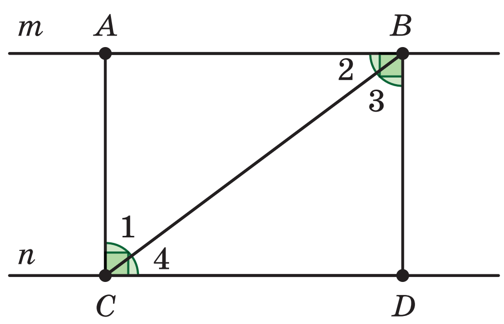
a) equal slanted lines have equal projections;

b) two slanted lines that are drawn from the same point have different projections to the third line. In this case the smallest slanted line has the largest projection.

3. Solve the problems. Explain your answer.

a) Given an isosceles triangle ABC ($AB = BC$) with sides 6.6 cm, 6.6 cm, 8 cm. What is the projection of the lateral side of the triangle to the base?
b) Given an equilateral triangle MNP with a side of 6 cm. What is the projection of the lateral side MN to the lateral side NP? What is the projection of the lateral side NP to the lateral side MN?
c) given a right triangle ABC ($\angle C = 90^0$) with sides $AB = 5$, $BC = 4$, $AC = 5$. What is the projection of the hypotenuse to the legs?

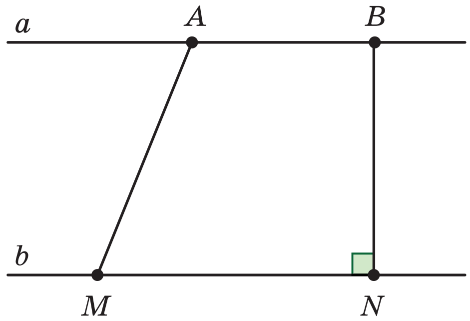
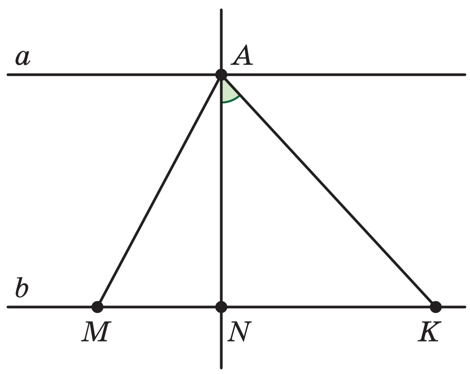
4. Given two parallel lines. Marat proved that all points of one line are equidistant from the other line. Comment his solution.

	**Given:** $m \parallel n$, $A, B \in m$, $C, D \in n$, $AC \perp n$, $BD \perp m$. **Prove:** $AC = BD$.
Proof: Since $AC \perp n, BD \perp m$, then $AC \parallel BD$, and $\angle 2 = \angle 4$, $\angle 1 = \angle 3$ (why?), therefore triangles ABC and BCD are congruent (why?), which means $AC = BD$.	

5. A traveller planned the route on the map. He knew that the distance between a point and a line is a perpendicular. Then he calculated a route along the closest slanted line to the perpendicular to see if it makes any difference in distance. What can you say about his route if *AB* is a perpendicular, *B* is the base of the perpendicular, and C is some point on the line. Make necessary constructions if *AB* =12 cm, *BC* = 2 cm.

6. Prove that the bisector of the triangle is between its median and altitude, drawn from the same vertex.

7. Solve the problems using ready made drawings.

	**Given:** $a \parallel b$, $A,\ B \in a$, $M,\ N \in b$. **Compare:** AM и BD.
	**Given:** $a \parallel b$, $AN \perp b$, $M,\ N, K \in b$. **Compare:** AM и NK.

8. Given triangle *PQR* with angle *R* is $30°$, side *PR* is 20 cm, and *QR* = 18 cm.

a) What is the distance between point Q and a side PR?
b) A straight line m parallel to a side QR was passes a vertex P. What is the distance between QR and a line m?

3.15 What do I know?

Fill in the table to revise your knowledge about properties and conditions of parallel lines and their application in problem solving.

<table>
<tr><td rowspan="3">Parallel lines</td><td>Conditions and properties of parallel lines
If alternate angles are equal then...
If the sum of consecutive angles is 180°, then...
If corresponding angles are equal then...
If lines are parallel then...
If lines are parallel, then alternate angles...
If lines are parallel, then corresponding angles...
If lines are parallel, then consecutive angles...</td></tr>
<tr><td>Comparing segments and angles of a triangle
The sum of interior angles of a triangle equals to...
The sum of exterior angles of a triangle equals to...
An exterior angle of a triangle is...
The largest angle is opposite to...
The smallest angle is opposite to...</td></tr>
<tr><td>Right triangles
If a leg and a hypotenuse...
If a leg and an acute angle...
If legs of right triangle...
In the right triangle a leg that is opposite to an angle...</td></tr>
</table>

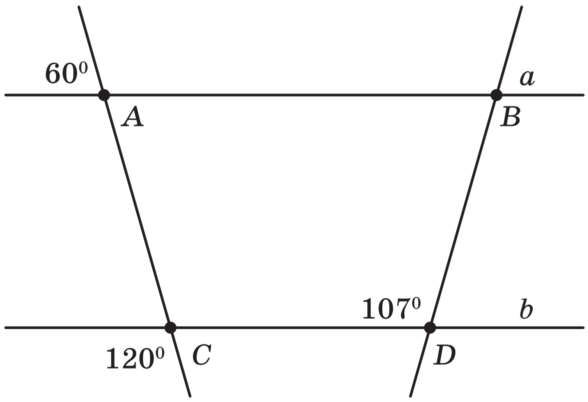
Questions that will help you to revise your learning.
Make sentences using the following words at least once:

- alternate angles;
- consecutive angles;
- corresponding angles;
- exterior angle of a triangle;
- sum of angles of a triangle;
- slanted line;
- perpendicular.

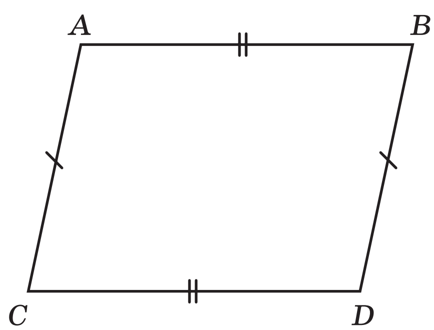
1. Is the following correct:

a) if the sum of one pair of alternate angles is respectively equal to another pair of alternate angles, then the lines are parallel;
b) there is a triangle with 45°, 38° and 100° angles;
c) the exterior and interior angles of the triangle are vertically opposite;
d) if the two sides of an isosceles triangle are 5 cm and 11 cm, then the third side of the triangle is 5 cm?

2. Solve the problems using ready made drawings:



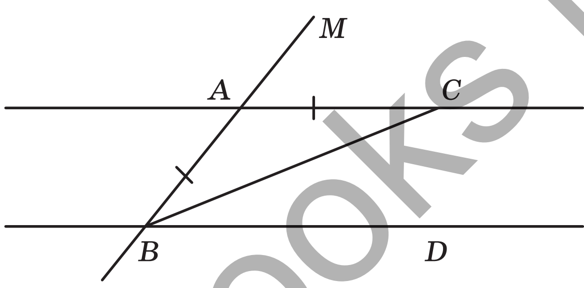
Find: α .



Given:

$AB = CD, \; BC = AD.$

Prove: $BC \parallel AD$.

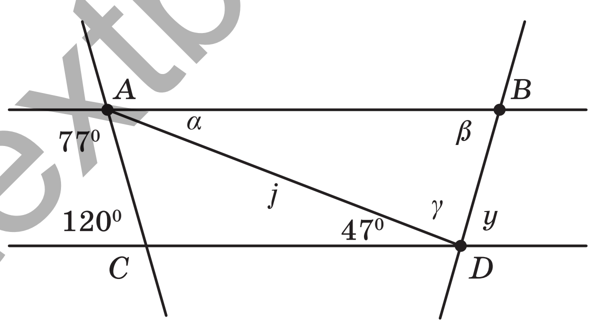


Given:

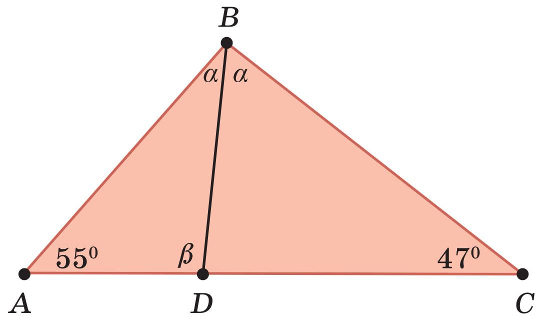
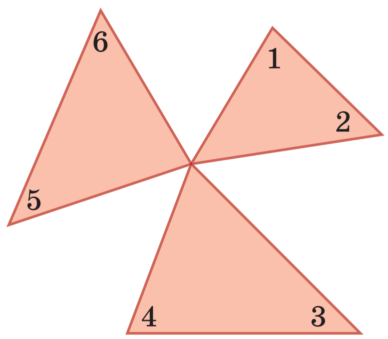
$AC \parallel BD,$

$AC \; = AB, \; \angle MAC \; = \; 40°$

Find: $\angle CBD$.



Find: $\alpha, \; \beta, \gamma$.

	**Find:** $\alpha,\ \beta$.
	**Find:** $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6$.

3. Arman shipped out on a boat taking a course of 45˚ angle to the coastline. He sailed 6 km then turned 70˚ in the opposite direction and after another 6 km he return to the port. What angle did the course have when Arman sailed to the port?

4. Draw a triangle *ABC* and mark point M inside the triangle. Compare perimeters of triangles *ABC* and *AMC*.

3.16 What do I know? Self assessment

1. Given lines *a, b, c,* when $a \perp b$, $b \perp c$. What statement is correct:

a) $a \perp c$;

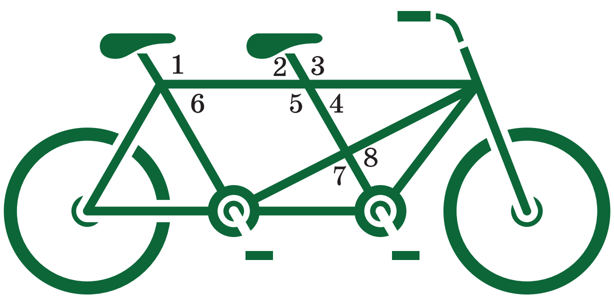
b) $a \parallel c$;

c) a, b and c and c have three common points;

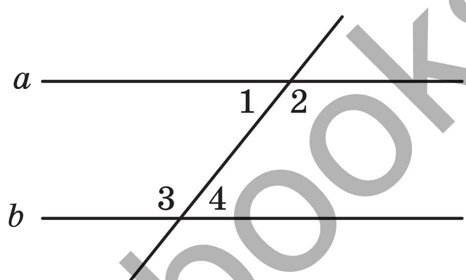
d) two of three lines a, b and c are parallel.

Explain your answer.

2. Work with the drawing and describe all the angles.

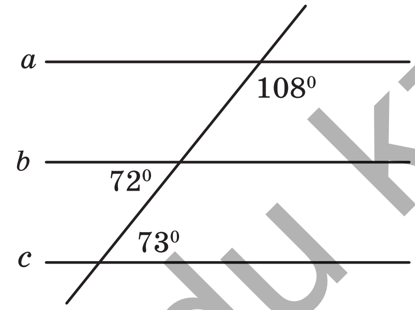


3. Lines *a* and *b* are parallel. What can you tell about angles 1 and 3?



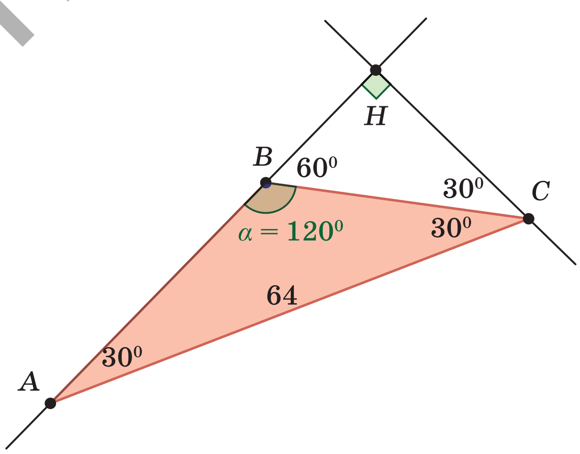
4. There are two lines and one transversal. We know three of them add up to 200°. Find all angles.

5. What lines of the given are parallel? Why?



6. Given isosceles triangle with 35° angle at the base. Find exterior angles of the triangle?

7. Given an isosceles triangle *ABC* with the base *AC* = 64 cm, and 60° exterior angle at the vertex *B*. What is the distance from point *C* of the line containing side *AB*?

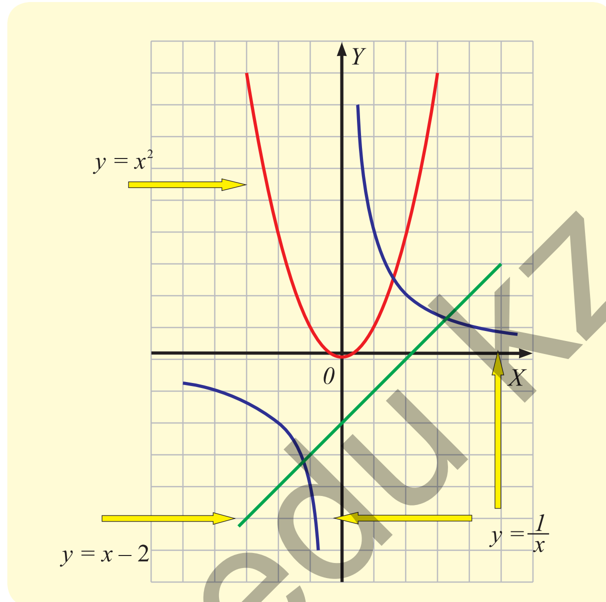


8. Angles of the triangle *ABC* are related as 1:6:8. Find angle *A* if side *BC* is the smallest.

4 Function and graph of a function

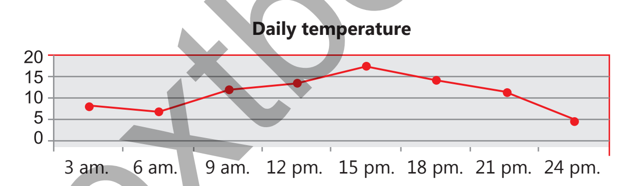
By the end of this unit, you will have learned:
- ✔ what is a function;
- ✔ how to define a function;
- ✔ what are domain and range of a function;

I will be able to:
- ✔ plot graphs of functions $y = kx + b,\ y = kx,\ y = ax^2,$ $y = ax^3,\ y = |x|;$
- ✔ use the properties of a function when solving problems.



Functions around us.

3 am.	6 am.	9 am.	12 pm.	15 pm.	18 pm.	21 pm.	24 am.
+9ºC	+8ºC	+12ºC	+14ºC	+17ºC	+15ºC	+12ºC	+6ºC



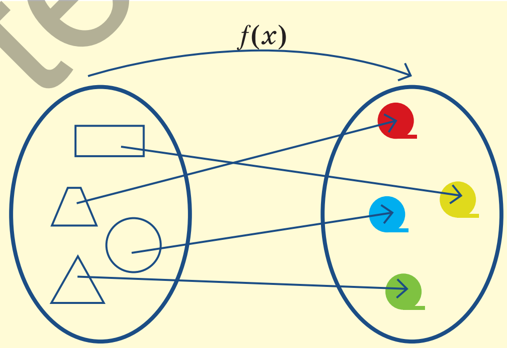
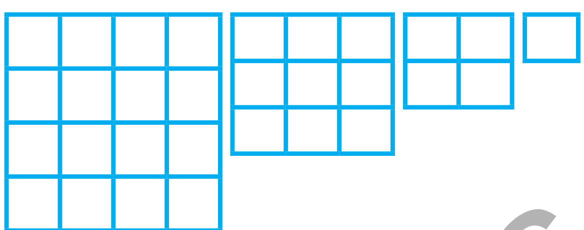


Figure	Colour
rectangle	
trapezoid	
circle	
triangle	

4.1 Function. Domain and range of a function

You have seen many situations when one value of a quantity depends on a value of another quantity. Let us take examples of such situations.

When you go to school, the time you get there depends on the speed you're moving at. Each value of speed v corresponds to a value of time t spent on the travel, i.e. the faster you go, the less time you spend on the travel, and the slower you are, the more time you need.



If you draw a square, you can see that its perimeter depends on the length of its side. Each value of the square side length *a* coorresponds to *a* single value of its perimeter *P*, i.e. when you increase the side of the square, the perimeter will increase, and vice versa, when you decrease the side, the perimeter will decrease as well.

The mass of an ice cream cup depends on its volume. Each value of the volume *V* of the ice cream corresponds to a single value of its mass *m*.

x	0	1	2
$3x + 2$	2	5	8

When you find the value of the polynomial $3x + 2$, each value of the variable x corresponds to a single value of this expression.

The abovementioned examples are about the relationship between two variables, where one of the values depends on another value. This relationship is called a functional relationship.

If values of a variable are arbitrary, it is called *an independent variable* or *argument*.

If values of a variable depend on an argument, it is called a *dependent variable* or *function*.

1. Specify independent variables (arguments) and dependent variables in the examples given above.

If each value of a variable x from the range of D by virtue of the law or rule f corresponds to a single value of a variable y from the range of E, the variable y is called a function of x.

Function notation

Dependent variable (function)

The set of values of a function ***form a range of a function.***

$y = f(x)$

The law or rule to find the values y corresponding to the values of x

Independent variable (argument)

The values taken by an argument ***are domain of a function.***

For the function $y = f(x)$ the domain is given as ***D(f)*** or ***D(y)***, and the range is — ***E(f)*** or ***E(y)***.

2. Function or not a function? Specify the relationships that define a function. Explain your answer. Find the domain and range for the obtained functions.

The area of a square is a function of the square side length.

The area of a circle defined by the formula $S = \pi r^2$, is a function of its radius r.

The area of a rectangle S with sides of 4 cm and x cm is a function of its sides.

3. Write the formula $f(x) = x + 5$.

a) Find values of y corresponding to each value of x and complete the table:

x	–3	–2	–1	0	1	2	3
y							

Can we say that this formula defines the function? Why? Explain your answer.

b) Find the domain and range of the given function.

4. The function is defined by the formula (*analytically*). Specify dependent and independent variables.

a) $y = x^2 - x$; b) $x = 2 + p^4$;

c) $u = \frac{1}{9}k - k^3$; d) $w = \frac{a}{a-2} + 3$.

Functions may be defined by:
- formula (analytically);
- table.

5. The function is defined by the table:

x	−4	−3	−2	−1	0	1	2	3
y	12	9	6	3	0	−3	−6	−9

Fill the gaps:

a) The argument (-2) corresponds to the value of the function equal to … .
b) If the value of the argument is 1, the value of the function will be … .
c) The value of the function is 9, if the value of the argument is … .
d) $-4 \rightarrow \ldots$; $f(2) = \ldots$; $\ldots \rightarrow 3$; $f(\ldots) = 9$.

Write a formula to define this function.

6. The relationship between variables *y* and *x* is defined by the table. Is this relationship a function? Why? Explain your answer.

x	1	3	5	8	9
y	3	3	5	5	0

x	2	4	0	4	2
y	0	2	4	6	8

The values of a function may be written by different ways: $-3 \rightarrow 9$ means that the value of an argument (-3) corresponds to the value of a function 9, $f(-3) = 9$ means the same.

Draw your own tables so one of them includes the functional relatinship.

4.2 Function. Domain and range of a function. Problem solving

1. Given a function $f(x)=\frac{6}{x}-2$ **is given. Find:**

a) $f(-3)$; **b)** $f(12)$; **c)** $f\left(\frac{3}{2}\right)$; **d)** $f\left(-\frac{12}{5}\right)$; **e)** $f(0{,}3)$.

2. Let us assume that $f(x)=2x+3$; $h(x)=2x^2$. **Find:**

a) $f(1)+h(-1)$; **b)** $f\left(\frac{1}{2}\right)-h(2)$; **c)** $f(0{,}1):h(0{,}1)$; **d)** $f(4)\cdot h(3)$.

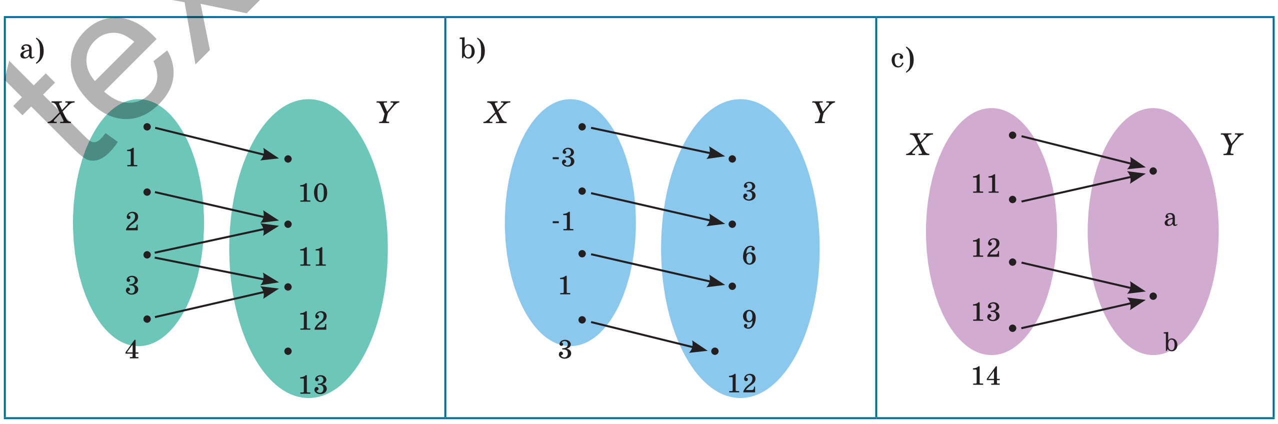
3. Draw a table of values for the funtion $y=\frac{20}{x-2}$, **if the argument x takes integer values and** $-5\le x\le 5$.

4. Translate the following statements into mathematical language:

a) the function takes the value of 5, if the value of the argument equals 3.
b) the value of a function with the argument 7 equals to the value of a function with the argument 4;
c) the function takes equal values, if the values of the argument are 8 and (-2);
d) the sum of the values of a function equals to 1, if the values of arguments are 6 and (-6).

> **Sometimes a function is represented as matching between the elements of two ranges of *X* and *Y*, when each element of the range of *X* corresponds to a single element of the range of *Y*.**

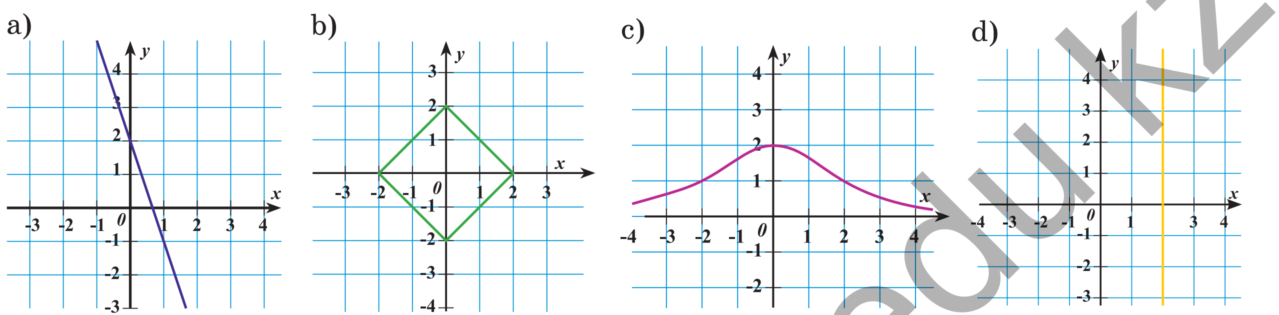
5. Work with a picture and answer the questions.
The matching between the elements of ranges of *X* and *Y*.



a) What are the similarities and differences between the given matches?
b) Which of these matches define the function? Why do you think so?
c) Find the domain and range for the matches defining the function.

The ***graph of a function*** is a set of points on a coordinate plane, the abscissas of which are equal to the values of the argument, and the ordinates of which correspond to the values of the function..

6. Are the lines given below graphs of functions? Why? Explain your answer.



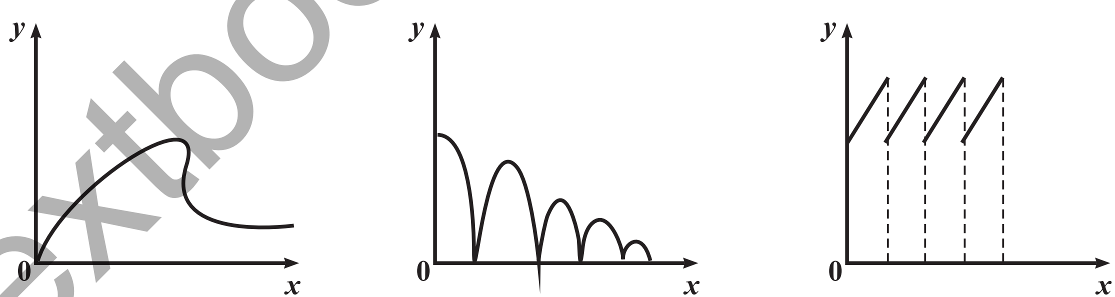
7. The relationship between two quantities is defined by the formula $f(x) = 4x - 2$.

a) Complete the table:

x	−3	−2	−1	0	1	2	3
y							

b) Use the table to plot the graph of this relationship. Does it define the function? Why?
c) If this relationship is a function, find its domain and range.

8. Match the graphs and situations given below.



a) the ball falls from a certain height to the floor (x - time, y - height of the ball above the floor);
b) grass is growing on the lawn, which is regularly mown (x - time, y - height of the grass);
c) apple is growing, but then it is picked and dried (x - time, y - mass of the apple).

Explain why these graphs are the graphs of functions.

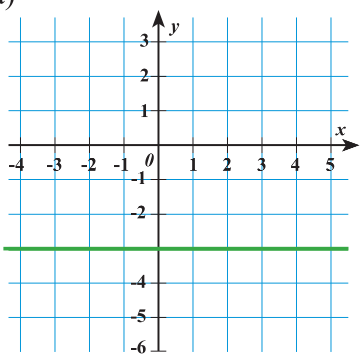
4.3 Linear function and its graph

Previously when you studied mathematics in grade 6, you had to deal with some types of functions, such as linear function. Let us recall that you already know about it and try to learn more.

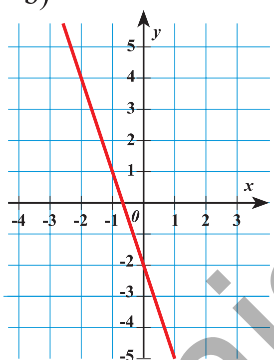
> The function $y = kx + b$, where x is an independent variable (argument), k and b are numbers, is called *a linear function.* The graph of the function is a straight line.

1. Are the lines below graphs of functions? What functions do these lines correspond to? Why do you think so? Explain your answer.

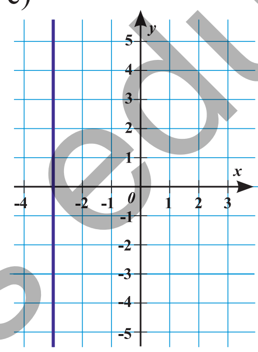
a)



b)



c)



2. Fill the table, use its data and plot the graphs of functions in a single coordinate system. Can you see the pattern? Make your conclusion.

x / $y = x + b$	-2	-1	0	1	2	3
$y = x$						
$y = x + 2$						
$y = x + 5$						
$y = x - 3$						

Summary:

- The graphs of the functions given in the table are…
- All the functions given in the table are defined…
- All the functions given in the table are…
- To construct a linear function, it is necessary to … points. (Why?)

3. Specify linear functions. Explain your answer.

a) $y = \frac{x}{7} - 5$; **b)** $y = \frac{7}{x} - 5$; **c)** $y = \frac{2x - 4}{2}$; **d)** $y = x^2 + 2$.

4. Complete the table and plot graphs of the functions. Find the coordinates of the points of intersection between the graph and coordinate axes.

a) $y = x + 3$;

x	0	2
y		

b) $y = -\frac{1}{3}x - 4$;

x	0	3
y		

c) $y = 0,2x - 2$;

x	0	5
y		

d) $y = 3x - 4$;

x	2	
y		-10

e) $y = \frac{1}{2}x - 2$;

x	4	
y		3

f) $y = -0,5x + 6$.

x	2	
y		-3

5. Use the results to complete the table and answer the questions.

Function	$y = x + 3$	$y = -\frac{1}{3}x - 4$	$y = 0,2x - 2$	$y = 3x - 4$	$y = \frac{1}{2}x - 2$	$y = -0,5x + 6$
k	**1**					
An angle between a straight line and positive axis *Ox* (obtuse or acute)	acute					

a) What is the relationship between the value of coefficient k and angle between a straight line and positive direction of Ox axis?

b) If $k = 0$, what is the angle between a straight line and positive direction of Ox axis?

c) What is the relationship between the coefficient b and position of the graph of a function?

REMEMBER!

Coefficient k of a function $y = kx + b$ is called a **slope**.

6. Draw a schematic graph of the function $y = kx + b$ graphically, if:

a) $k > 0,\ b = 0$;

b) $k < 0,\ b > 0$;

c) $k < 0,\ b = 0$;

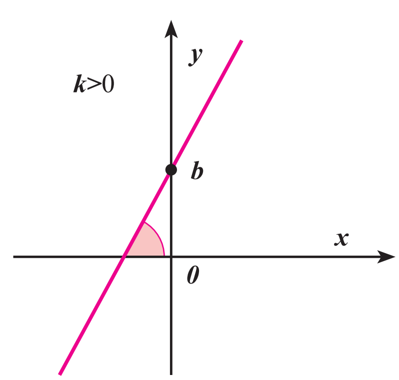
d) $k = 0,\ b > 0$.

What other cases are possible?

7. Given a box with candies. The mass of each candy is 3 g. The mass of the empty box is 150 g. How much does the box weigh with n number of candies? Plot the graph of a function $m = f(n)$, where m is the mass of the box with candies. Find the domain and range of the function.

Example:

If $k > 0,\ b > 0$, the graph of the function will be:



4.4 Linear function and its graph. Problem solving

1. Find out whether the graph of a function $y = 1{,}5x + 10$, intersects the following points without plotting a graph:

a) $A(4; 15)$; $C(4; -4)$; $E(-100; -140)$;

b) $B(-2; 7)$; $D(80; 160)$; $F(0; 10)$.

2. Plot the graph of a function:

a) $y = -2x + 1$, where $-3 \le x \le 3$;

b) $y = -2x + 1$, where $x \ge 0$;

c) $y = -2x + 1$, where $x \le -1$;

d) $y = -2x + 1$, where $x \in \{-4; -3; -2; -1; 0; 1; 2; 3; 4\}$.

Find out if there is a point on the graphs with the same (opposite) values of abscissa and ordinate?

3. Arman says that he can plot a graph of a linear function, if he knows the slope and point belonging to this graph. How can he do this? Explain your answer.

4. The line $y = 3x + b$ intersects the point A. Find b, if you know the coordinates of the point A.

a) $(1; 3)$;

b) $(-1; 1)$;

c) $(0; 2)$.

Write the equations of the lines. Find the coordinates of the points of intersection between lines and coordinate axes without plooting a graph.

5. The graph of a function $y = kx + b$ intersects points A and B. Find the values of k and b, if:

a) $A(0; 2)$ and $B(3; 6)$;

b) $A(0; 3)$ and $B(4; 5)$;

c) $A(0; -4)$ and $B(3; -4)$.

6. Plot the following functions $y = 3x + 3$ and $y = 3x$ and complete the table. What are the differences and similarities between these functions?

	$y = 3x + 3$	$y = 3x$
Domain		
Range		
Points of intersection between a graph and coordinate axes		

REMEMBER!

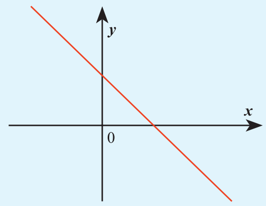
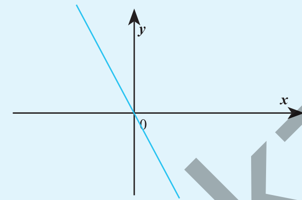
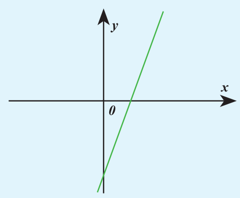
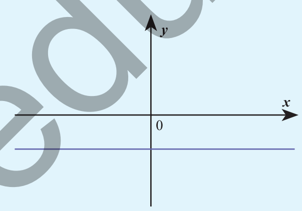
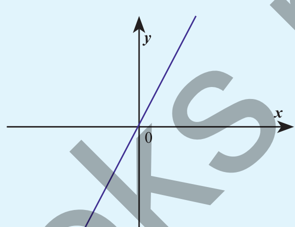
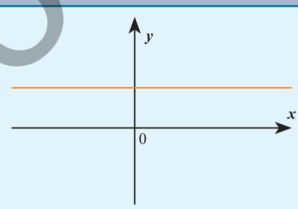
The function $y = kx$, where x is an independent value or argument, k is a non-zero number, is called a direct proportionality. The graph of this function is a line intersecting the origin.

The function $y = kx$ is a special case of the function $y = kx + b$.

7. Match the values that may be taken by *k* and *b* coefficients and graphs of functions. Which of the figure shows the graph of direct proportionality?

Coefficients *k* and *b*

Graphs of functions

1.	$k > 0,\ b > 0$	5.	$k > 0,\ b = 0$
А		Д	
2.	$k > 0,\ b < 0$	6.	$k < 0,\ b = 0$
Б		Е	
3.	$k < 0,\ b > 0$	7.	$k = 0,\ b > 0$
В			
4.	$k < 0,\ b < 0$	8.	$k = 0,\ b < 0$
Г			

9. Marat says that functions $y = \frac{6x^2}{x}$ **and** $y = 6x$ **are the same. Is Marat right? Explain your answer. Plot the graph of the function** $y = \frac{6x^2}{x}$.

4.5 Linear function and its graph. Problem solving

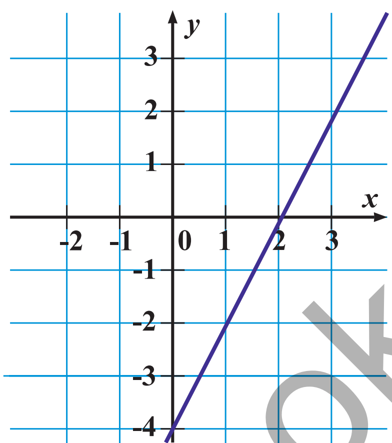
We have considered the questions about how to plot a linear function, if it is defined analytically, or if we know its coefficients. Do you think it is possible to define this function analytically, if we know its graph? Let us talk about this in more detail.

REMEMBER!

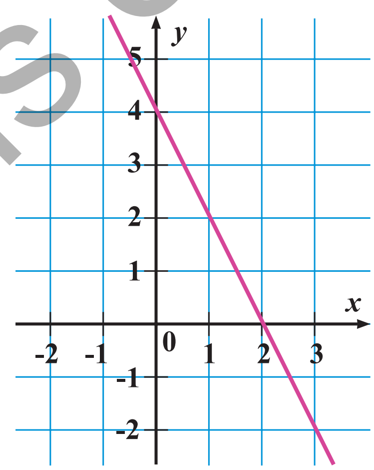
The slope k shows the angle of inclination between the straight line and positive direction of Ox axis. The number b is the ordinate of the point of intersection between the line and axis of ordinate.

1. Which figure shows the graph of the following function $y = 2x + 4$? Why do you think so? Use your knowledge about coefficients k and b.

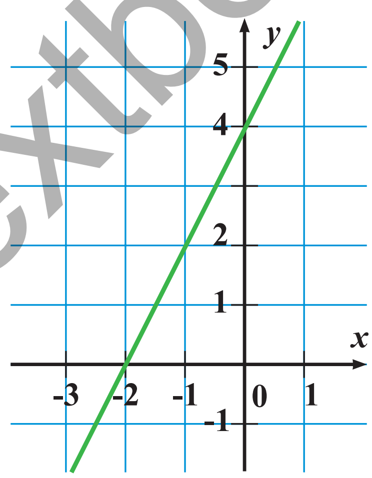
a)



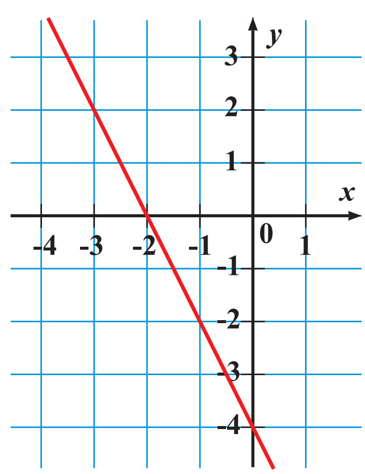
b)



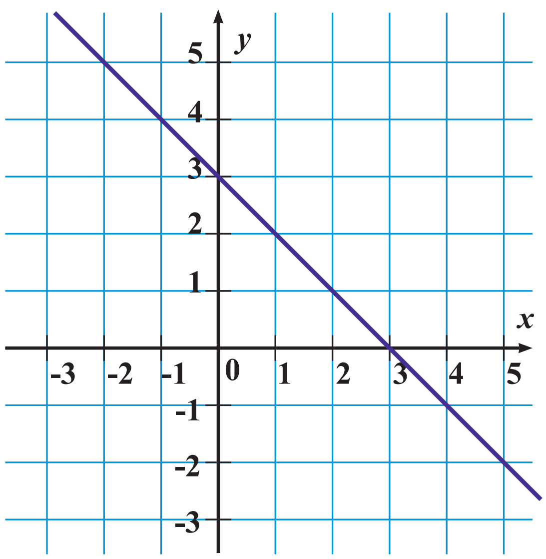
c)



d)



2. Define a function analytically.



Solution: since the graph of the function is a straight line, it can be defined as $y = kx + b$.

1. Define the coefficients k and b.
$b = 3$, because Oy intersects the point (0; 3), so the function can be written as: $y = kx + 3$.

2. $k < 0$, because the angle of inclination is obtuse. Choose a random point belonging to the graph of the function and find the value of k. For example, point $A(3; 0)$ and we will get $0 = 3k + 3 \Rightarrow k = -1$.

Then, the function may be defined analytically by the following equation: $y = -x + 3$

3. Define the functions analytically, the graphs of which are given in activity 1.

4. Plot the graphs of the following functions on one coordinate plane:

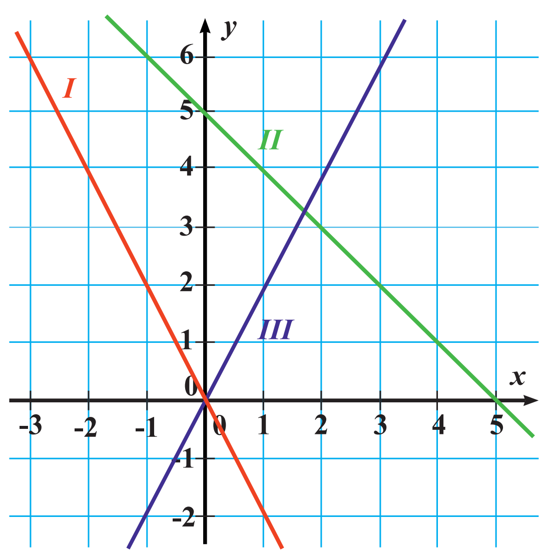
a) $y = 3x$;   b) $y = 3x + 4$;   c) $y = 3x - 5$.

Compare the positions of the graphs of functions. How can we get the graphs of functions $y = 3x + 4$ and $y = 3x - 5$ from the graph of the function $y = 3x$.
Try to make an algorithm of how to plot the graph of the function $y = kx + b$ using the graph of the function $y = kx$.

5. Use a pattern to plot the graph of a linear function using two types of technique:

a) $f(x) = -3x + 2$;   b) $f(x) = \frac{1}{2}x - 3$.

6. Find a function. The figure shows the graphs of the functions: $y = -2x$, $y = 2x$, $y = -x + 5$. Specify the formulas that correspond to each of the graphs.

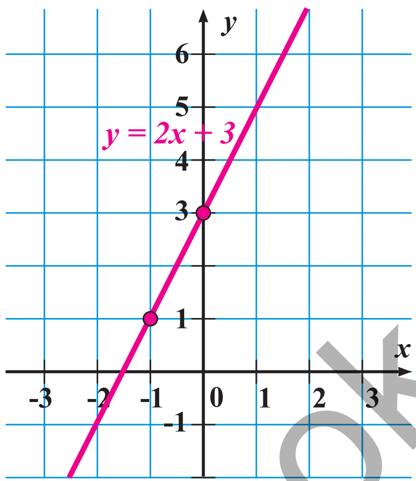


TWO WAYS TO PLOT A LINEAR FUNCTION:

On two points

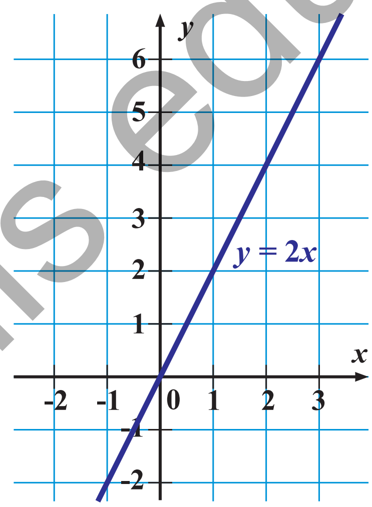
1. Draw a table;
2. Construct a line on two points.

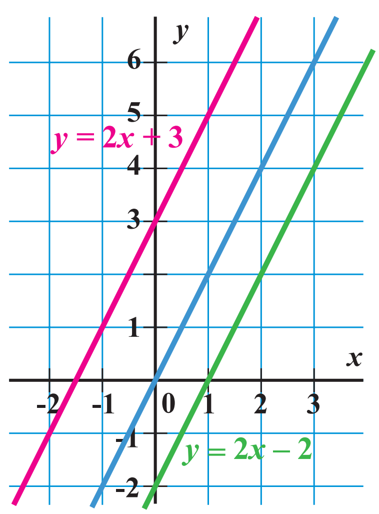
x	-1	0
y	1	3



Using a parallel transfer

1. Plot a graph of direct proportionality;
2. Move a graph along axis Oy by b units with use of parallel transfer.





4.6 Relative position of the graphs of linear functions

1. How to get the graph of the function $y = 3x - 4$ from the graph of the function $y = 3x$? And how to get the following graph of the function $y = 3x + 2$? Plot these graphs on one coordinate system. What can you tell about their relative position?

2. Plot the following graphs of functions on one coordinate system $y = \frac{2}{3}x - 2$ and $y = -x + 3$. What can you tell about their relative position?

3. Find the coordinates of the point of intersection without plotting the graph of a function:

a) $y = 7{,}9x - 6{,}3$ and $y = -6{,}3x + 7{,}9$; b) $y = 2{,}6x - 5$ and $y = 2{,}6x + 5$.

4. Use the activities 1-3 and match them with the cases below. Fill the gaps.

Linear functions	Algebraic conditions	Geometric conclusion
$y = k_1x + b_1$ $y = k_2x + b_2$	$k_1 = k_2,\ b_1 \neq b_2$	Straight lines $\boldsymbol{y = k_1x + b_1}$ and $\boldsymbol{y = k_2x + b_2}$ are parallel
	$k_1 \square k_2$	Straight lines $\boldsymbol{y = k_1x + b_1}$ and $\boldsymbol{y = k_2x + b_2}$ intersect
	$\boldsymbol{k_1 \ldots k_2,\ b_1 \ldots b_2}$	Straight lines $\boldsymbol{y = k_1x + b_1}$ and $\boldsymbol{y = k_2x + b_2}$ coincide

5. Write an equation of a line parallel to the line $y = -5x$ and intersecting the axis *Oy* at the point:

a) $(0; 2)$; b) $(0; -3)$; c) $\left(0; \frac{2}{7}\right)$.

6. Define the direct proportionality by the formula, if:

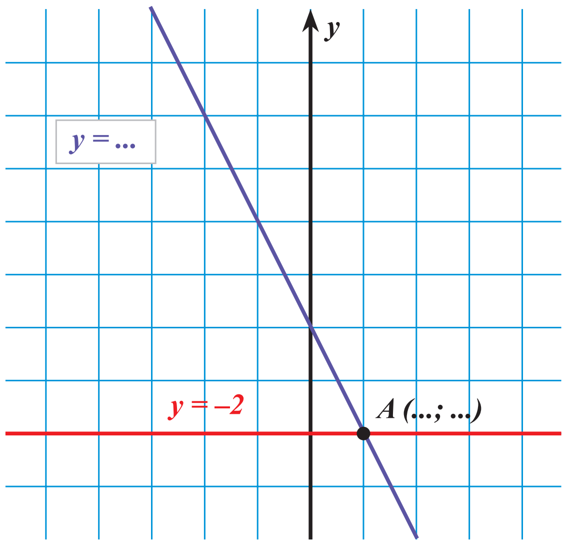
a) its graph and the graph of a function $y = 4x - 3$ are parallel;
b) its graph intersects point $A(1.3; -5)$.

7. Determine whether lines intersect or not, without plotting a graph. How can you do this work? Explain your answer.

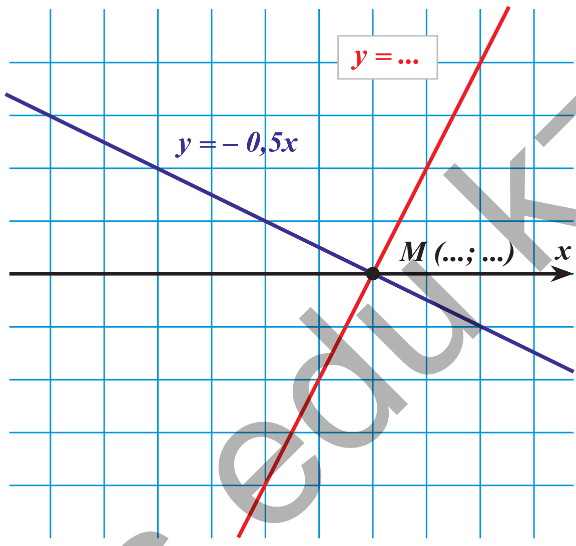
a) $y = 1{,}5x - 6$ and $y = 6x - 1{,}5$; b) $y = \frac{3}{4}x + 7$ and $y = \frac{3}{4}x - 4$?

8. Supplement the drawings with an image of axis *Oy* or *Ox* so that the line is a graph of the specified function. Fill the gaps with the formula of the second function. Specify the coordinates of the point of intersection between graphs of these functions and other designated points (unit segment - 1 cell).

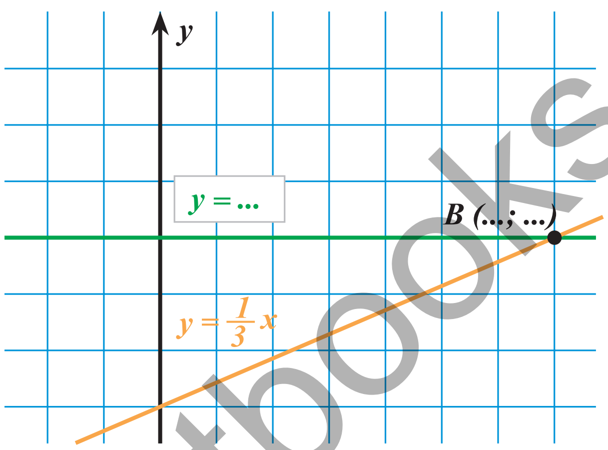
a)



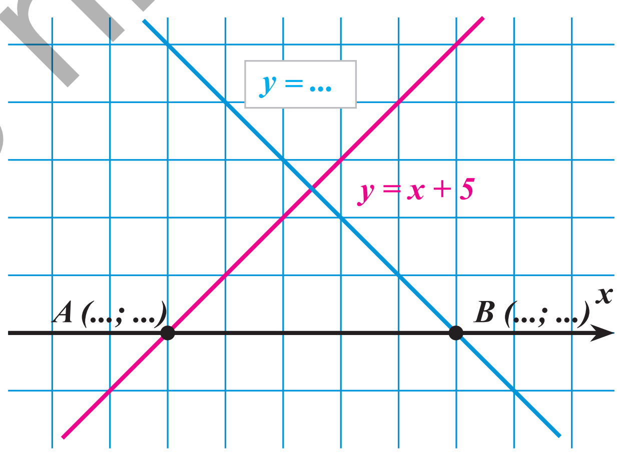
b)



c)



d)



4.7 Solving a system of linear equations with two unknown variables by a graphical method.

We can use knowledge of functions and their graphs when solving a large number of mathematical problems, and in particular, when solving equation systems. Previously we considered how to solve systems of linear equations with two variables by the method of substitution and algebraic addition. Let us now consider how we can solve such systems graphically.

> The equation of the following form $ax + by = c$, with variables x and y, is called a linear equation with two unknown variables

1. Dinara provided a solution of a linear equation with two unknown variables by a graphical method. Comment on the solution. Did she do everything right?

> $ax + by = c / : b \neq 0;\quad \frac{ax}{b} + y = \frac{c}{b};\quad y = \frac{c}{b} - \frac{ax}{b}.$ It means that the graph of a linear equation with two unknown variables is a straight line, and the solution of this equation is an ordered pair of numbers (x, y) - that are coordinates of the points belonging to this line.

2. Plot the graph of the following equations:

a) $0x + 5y = 20;$ **b)** $-7x + 0y = 21;$ **c)** $3x + y = 5;$ **d)** $5x + 6y = -11.$

Since the system of linear equations with two unknown variables has the following form:

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

the graphical solution of this system will be an ordered pair of numbers, which satisfies each equation of the system. This pair of numbers is the coordinates of the common points of the lines.

3. Help Sabina to make an algorithm to solve the system of linear equations with two unknown variables by a graphical method.

> **The algorithm for solving a system of equations by a graphical method.**
> **1. Draw a table of values for each function.**
> **2. Identify the number of solutions.**
> **3. Plot the graphs of functions on one coordinate plane.**
> **4. Bring both of equations to the form** $y = kx + b$
> 5. Write down the answer.

4. Solve the systems of equations by a graphical method:

a) $\begin{cases} y + x = 3, \\ 2x - y + 6 = 0; \end{cases}$ **b)** $\begin{cases} 4x - 2y = 10, \\ y - 2x = -3; \end{cases}$ **c)** $\begin{cases} 3y - 9 = 3x, \\ 5y - 15 = 5x. \end{cases}$

5. Look at the solutions of systems of equations from the previous activity. What pattern do you see? Make a conclusion about the number of system solutions.

Summary:

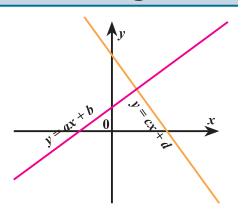
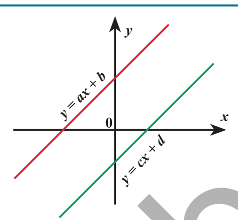
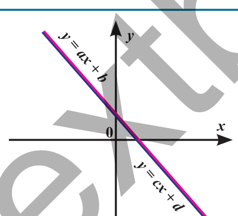
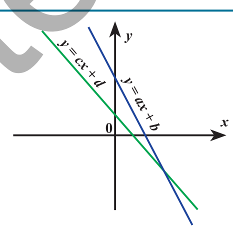
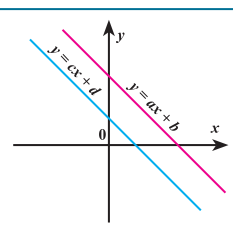
a) If the slopes of linear functions are different, the system has ... ;
b) if the slopes are the same, the system has ... ;
c) if both slopes and free terms are the same, the system has ... ;

Is it possible to determine the number of solutions without plotting graphs? Explain your answer.

6. Use the results of activities 4 and 5 and complete the table:

Straight lines	Common points	Solution of a system has	System is
	one common point		consistent and defined
		does not have solutions	inconsistent
			consistent and undefined

7. Match:

	Geometric image		Algebraic conditions		Geometric summary		Examples
I		А	$a \neq c,\ b = d$	1	parallel	а	$y = 2x - 3$ and $y = 2x + 4$
II		Б	$a = c,\ b \neq d$	2	intersect	б	$y = 3x + 1$ and $2y = 6x + 2$
III		В	$a = c,\ b = d$	3	intersect	в	$y = -\frac{2}{3}x + 5$ and $y = -\frac{3}{2}x + 1$
IV		Г	$a \neq c,\ b \neq d$	4	coincide	г	$y = x + 2$ and $y = -x + 2$
V		Д	$a = c,\ b = d$	5	parallel	д	$y = 5 - x$ и $y = 2 - x$

4.8 Solving a system of linear equations with two unknown variables by a graphical method. Problem solving

1. Solve the systems of equations by a graphical method:

a) $\begin{cases} 3x+y=2, \\ 3x+y=1; \end{cases}$ b) $\begin{cases} x+y=6, \\ y+3x=7; \end{cases}$ c) $\begin{cases} 3y-9=3x, \\ 5y-15=5x. \end{cases}$

2. Write a system of equations with two unknown variables so that the solution of this system is a pair of numbers:

a) $(0;5)$; b) $(-2;-1)$;

c) $(-1;\ 2)$.

3. Choose the second equation for each of the following equations so that the resulting system has only one solution:

a) $2x-3y=6$; b) $4x-5y=2$;

c) $6x+5y=8$.

4. Choose the second equation for each of the following equations so that the resulting system has a single solution:

a) $2x+5y=4$; b) $6x+9y=-7$;

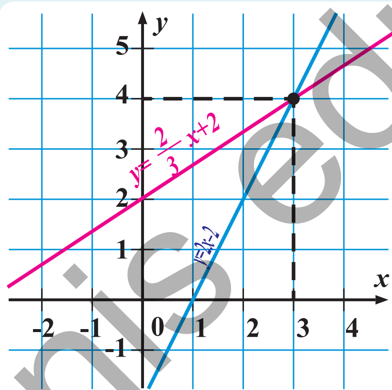
c) $\frac{1}{3}x-\frac{2}{5}y=9$.

5. Choose the second equation for each of the following equations so that the resulting system has no solutions:

a) $4x+y=6$; b) $3x-2y=5$;

c) $x-3y=6$.

REMEMBER!

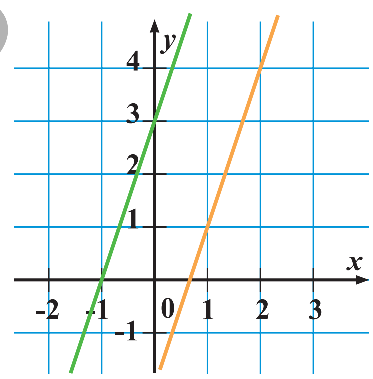


$k_1 \neq k_2$

One solution

$\begin{cases} y=2x-2, \\ y=\frac{2}{3}x+2. \end{cases}$

Answer: (3; 4).



$k_1 = k_2$
$b_1 \neq b_2$

No solutions

$\begin{cases} y=3x+3, \\ y=3x-2. \end{cases}$

Answer: no solutions.

$k_1 = k_2$
$b_1 = b_2$

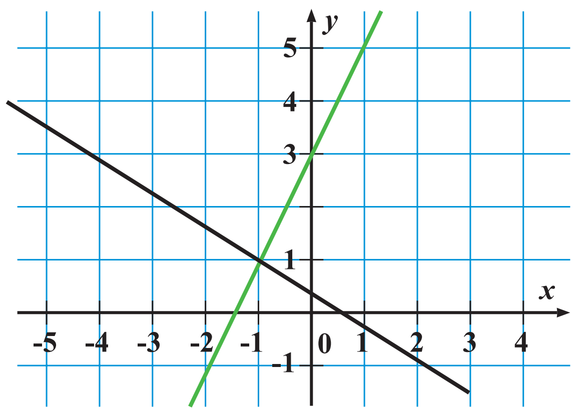
Infinite number of solutions

$\begin{cases} y=\frac{1}{3}x+3, \\ y=\frac{1}{3}x+3. \end{cases}$

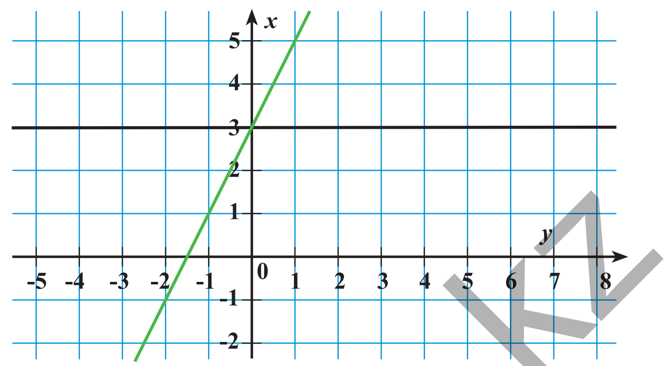
Answer: $\left(t;\ \frac{1}{3}t+3\right)$, $t \in R$.

6. Write a system of equations describing graphs given below.

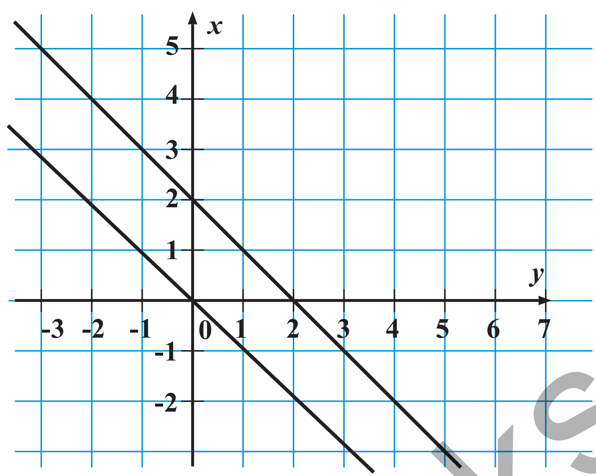
a)



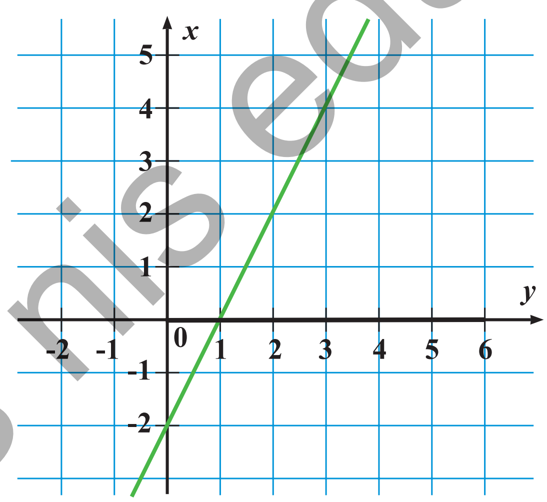
b)



c)



d)



7. Solve the system of equations by a graphical method, if we know that the first equation of the system is corrent equality at $x=2$ and $y=3$.

$$\begin{cases} 3x+ay=9, \\ 2x-3y=6 \end{cases}$$

8. Find the values of *p* parameter, at which the system of equations has only one solution.

$$\begin{cases} 3x+2y=10, \\ py-5x=15 \end{cases}$$

9. Find the values of *q* parameter, at which the system of equations has the infinite number of solutions.

$$\begin{cases} x+2y=1, \\ 3y+qx=1,5. \end{cases}$$

4.9 Function of the form $y=ax^2$

Besides linear functions, there are many other functions that you will learn throughout the school course of mathematics. Now we are going to talk about functions of the form $y = ax^2$.

1. Given a square with side *x*. Complete the table showing the relationship between the area of the square S and its side length *x*.

x	1	2	3	4	5
$S(x)$					

Write the formula for the relationship between the square area and its side length. What changes can you mention in the square area depending on the side length? Is this relationship a function? Why? Explain your answer.

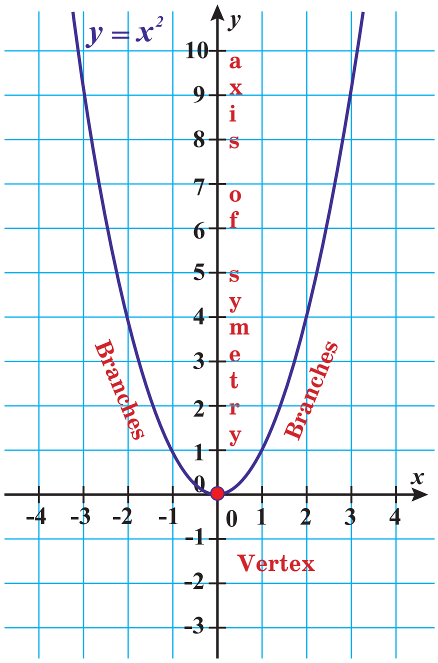
2. Conduct a research according to the plan.

The relationship between variables y and x can be expressed by the formula $y = x^2$.

1. Complete the table of values:

x	−3	−2	−1	0	1	2	3
y							

2. Use the table, construct points on a coordinate plane and connect them with a free line. Is this relationship a function? Explain your answer. The graph you plot has a special name – a parabola, and the relationships given above define a quadratic function.



> Functions of the form $y=ax^2$, where x is an independent variable or argument, y is a dependent variable, and a is a number ($a \neq 0$), is called a quadratic function. The graph is a parabola.

3. "Passport of a function". Fill the gaps.

$y = x^2$	
Domain	$D(y) = (-\infty; \square)$
Range	$E(y) = \ldots;$
Points of intersection between a graph of a function and coordinate axes	$Ox: y = 0,\ x = \square;$ $Oy: x = 0,\ y = \square.$

4. Plot the graph of a function $y=-x^2$. Specify a domain and range of the function, and the points of intersection between the graph and coordnate axes.

5. Fnid out whether the following points belong to the graph of a function $y=x^2$ without plotting the graph of a function:

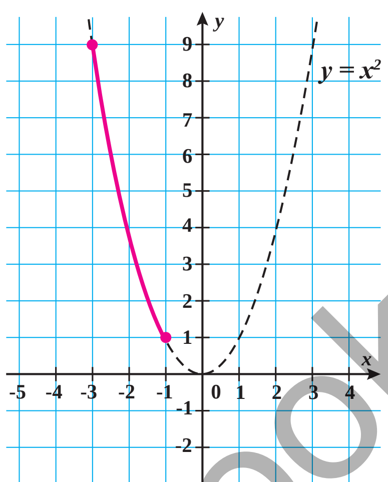
a) $A(-4;16)$; b) $B(-2;-4)$; c) $C\left(\frac{1}{2}; 0,25\right)$; d) $D(3;8)$?

6. Fnid out whether the following points belong to the graph of a function $y=x^2$ without plotting the graph of a function:

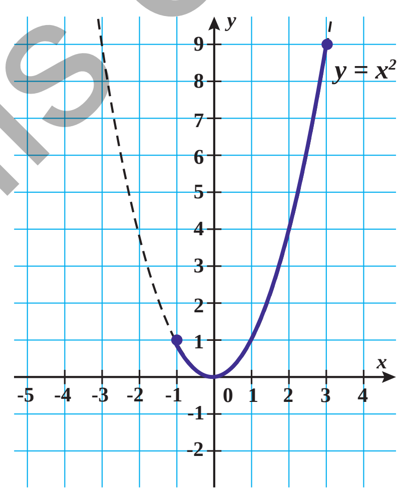
a) $K(-3;9)$; b) $P(-1;-1)$; c) $L\left(\frac{2}{3}; -\frac{4}{9}\right)$; d) $M(-3;-6)$?

7. Use the selected part of the graph of a function, find the maximum and minimum value of the function. Explain your answer:

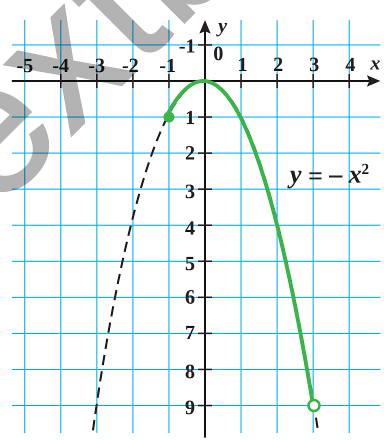
a)



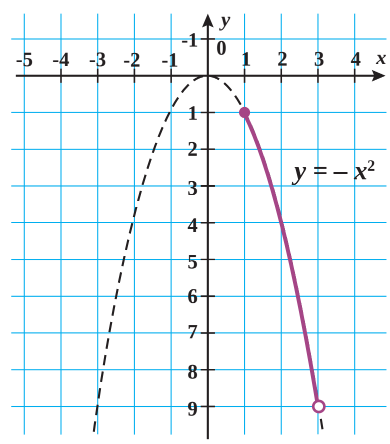
b)



c)



d)



8. Given a function $y = x^2$. Compare $y(u)$ and $y(v)$ without calculating the values of functions at specified points, if:

a) $u = 1,5;\ v = 1,7;$

b) $u = 0,1;\ v = 0,08;$

c) $u = -10,13;\ v = -10,1;$

d) $u = -3,4;\ v = -3,8;$

Explain your answer.

9. Plot the graph of a function $y = 2x^2$. Use the graph to find:

a) values of the function when the argument takes values equal to (−2); 0; и 1;
b) values of an argument when the function takes values equal to 0; 2; 8;
c) maximum and minimum values of the function on the section $[-2;\ 1]$;
d) values of an argument, at which $2 \le y \le 8$.

10. Plot the graph of a function $y = -\frac{1}{2}x^2$. Use the graph to find:

a) values of the function when the argument takes values equal to (−2); 0; 4;
b) values of an argument when the function takes values equal to (−4,5); (−2); 0;
c) maximum and minimum values of the function on the section $[-1;\ 3]$;
d) values of an argument, at which $-4,5 \le y \le -2$.

11. Compare the solutions of activities 9 and 10. What interesting things have you noticed? Make a conclusion:

If $a > 0$, the function has a minimum value, and does not have a maximum value.
If $a < 0$, then ____ .
Is this output correct for all quadratic functions?

4.10 Function of the form $y=ax^3$ and $y=|x|$

Given a cube with edge x. Complete the table showing the relationship between the volume of the cube V and length of the edge x. $y = ax^3$.

1. Given a cube with edge x. Complete the table showing the relationship between the volume of the cube V and length of the edge x.

x	1	2	3	4	5
$V(x)$					

Write the formula for the relationship between the cube volume and its side length. What changes can you mention in the volume depending on its side length? Is this relationship a function? Why? Explain your answer.

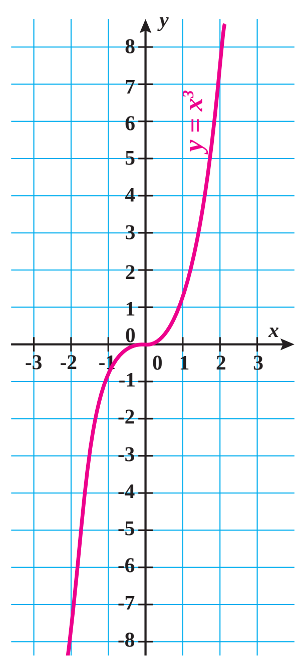
2. Conduct a research according to the plan.

The relationship between variables y and x can be expressed by the formula $y = x^3$.

1. Complete the table of values:

x	-2	-1	0	1	2
y					

2. Use the table, construct points on a coordinate plane and connect them with a free line. Is this relationship a function? Explain your answer.



The graph you plot is called a cubic parabola.

> **The function $y=ax^3$, where x is an independent variable or argument, y is a dependent variable, and a is a number ($a \neq 0$), is called *a cubic function*. The graph is *a cubic parabola*.**

3. "Passport of a function". Fill the gaps.

$y = x^3$	
Domain	$D(y) = (-\infty; \square)$
Range	$E(y) = \dots$;
Points of intersection between a graph of a function and coordinate axes	$Ox: y = 0,\ x = \square$; $Oy: x = 0,\ y = \square$.

4. Use the graph of a function $y = x^3$, to find:

a) values of the function when the argument takes values equal to (−1); 0; 1,3; 2;
b) values of an argument when the function takes values equal to (−3,5); (−1); 0,9; 7;
c) range of x, at which the value of the function is less than 1; more than (-3); more than 1, but less than 3.

5. Plot the graph of a function $y = -x^3$. Specify the domain and range of the function, and the points of intersection between the graph and coordinate axes.

6. Plot the graphs of functions on one coordinate system (unit segment is one cell) $f(x) = -x^2$, $y = 3x - 4$. Find the abscissas of intersecting points.

7. Plot the graphs of functions on one coordinate system (unit segment is one cell) $f(x) = x^3$, $y = -2x + 3$ and find the abscissas of intersecting points.

8. Plot the graph of a function $y = -2x^3$. CUse the graph to find:
a) maximum and minimum values of the function on the segment $[-1; 1]$;
b) value of an argument, at which $0 \le y \le 2$.

9. Plot the graph of a function $y = \frac{1}{2}x^3$. Use the graph to find:
a) maximum and minimum values of the function on the segment $[-1; 2]$;
b) value of an argument, at which $-4 \le y \le 0$.

10. Given a function $y = f(x)$, where $f(x) = \begin{cases} x^2, & \text{if} \quad -2 \le x \le 0; \\ 3x, & \text{if} \quad 0 < x \le 2; \\ 6, & \text{if} \quad 2 < x < 5. \end{cases}$

a) Calculate $f(-2)$, $f(-1)$, $f(0)$, $f(1{,}5)$, $f(3)$, $f(4)$;
b) Plot the graph of a function $y = f(x)$;
c) Find a domain, range and points of intersection between the graph of the function and coordinate axes.

11. Plot the graph of a function $y = |x|$. Find a domain, range and points of intersection between the graph of the function and coordinate axes.

4.11 Function of the form $y=\frac{k}{x}$

1. Think and answer the following questions:

a) Which values are called directly proportional and which are inversely proportional?
b) Give some examples of directly proportional and inversely proportional values.
c) What is the formula that makes connection between directly proportional and inversely proportional values?

2. Specify directly proportional and inversely proportional relationships. Explain your answer.

a) $y=2x$; b) $y=-2x$; c) $y=2x^2$; d) $y=-2x^3$; e) $y=\frac{2}{x}$;

f) $y=\frac{x}{2}$; g) $y=-\frac{2}{x}$; h) $y=\frac{x^2}{2}$; i) $y=-\frac{x}{2}$; j) $y=-\frac{x^3}{2}$.

3. Translate the problem condition into mathematical language and write down the formula for the relationship between the given values.

a) Temirlan traveled 60 m by bicycle in *t* min. What is his speed?

b) The area of a rectangle with a m of length equals 50 m2. What is the width of the rectangle?

Is this a functional relationship? Why? Explain your answer. Give your examples of similar relationships.

4. Conduct a research according to the plan.

The relationship between variables *y* and *x* can be expressed by the formula $y=\frac{6}{x}$.

1. Complete the table of values:

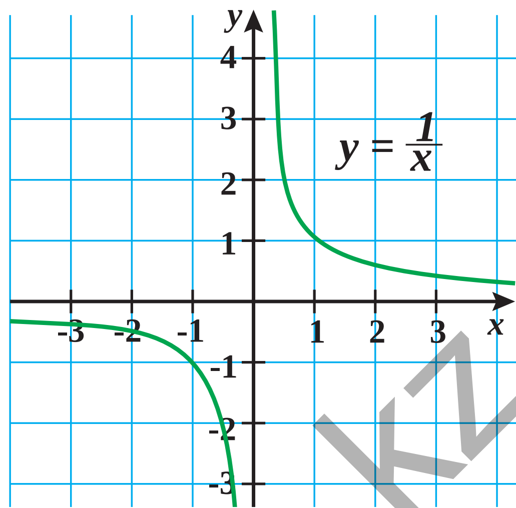
x	–6	–3	–2	–1	1	2	3	6
y								

2. Use the table, construct points on a coordinate plane and connect them with a free line. Is this relationship a function? Explain your answer.

The function $y=\frac{k}{x}$, where x is an independent variable or argument, *y* is a dependent variable, and *k* is a number ($k\neq0$), is called an inverse proportionality. The graph is a hyperbola.

5. "Passport of a function". Fill the gaps.

$y=\frac{1}{x}$	
Domain	$D(y) = (-\infty; \square);$
Range	$E(y) = \ldots ;$
Points of intersection between a graph of a function and coordinate axes	$Ox:\ y=0,\ x=\square;$ $Oy: x=0,\ y=\square.$



6. Define an inverse proportionality by the formula, if its graph intersects the following point:

a) $A(2;3,5)$; b) $N(\frac{3}{4};-6\frac{2}{3})$.

Specify four more points belonging to this graph of the function.

7. Plot the graphs of functions in one system coordinate $y=\frac{4}{x}$ and $y=-\frac{4}{x}$. State a domain and range of the functions. What are the similarities and differences of the functions? What is the relationship between the graph of the function and coefficient k?

8. State the coordinate quadrant that includes the following function without plotting a graph:

a) $y=\frac{5}{x}$; b) $y=-\frac{10}{x}$;

c) $y=\frac{1}{6x}$; d) $y=-\frac{0,5}{x}$.

REMEMBER!

The graph of a function $y=\frac{k}{x}$ at which:

- $k>0$, is in the I and III coordinate quadrants;
- $k<0$, is in the II and IV coordinate quadrants.

9. What will be the graph of the function $y=\frac{k}{x}$ if $k=0$?

10. Draw schematic graphs of the functions $y=mx+l$ and $y=\frac{k}{x}$ so that they have:

a) one point of intersection;
b) two points of intersection;
c) three points of intersection.

Write the equations of the obtained graphs of functions.

4.12 What have I learned?

When you finish, you will repeat what you learned about functions and their graphs.

A function and graph of a function

A *function* is …

A variable x is …

A variable y ... — .....

A function can be defined …

A graph of a function is a set of points …

A range of an argument is …

A range of a function is …

A linear function and its graph

A linear function is a function of the form

Coefficient k is called …

If $k > 0$, the straight line $y = kx + b$ forms a … with a positive direction of Ox axis, and if $k < 0$ — ... .

A function of the form $y = kx$, where x — ... , k — ... , is called ... .

Relative position of graphs of linear functions

Straight lines $y = k_1x + b_1$ and $y = k_2x + b_2$ *are parallel, if* ... .

Straight lines $y = k_1x + b_1$ and $y = k_2x + b_2$ *intersect, if* ... .

Straight lines $y = k_1x + b_1$ and $y = k_2x + b_2$ *coincide, if* ... .

Function of the form $y=ax^2$, $y=ax^3$ $(a\neq0)$

The quadratic function is a function of the form ... , where ... .

The graph of a quadratic function is ... .

The function of the form $y = ax^n$, where … is called … .

The graph of the function $y = ax^3$ is called ... .

Function of the form $y = \frac{k}{x}$ $(k \neq 0)$

The function of the form $y = \frac{k}{x}$, where … is called … .

The graph of the function $y = \frac{k}{x}$ is ... .

Questions to help you repeat the previously studied materials

Write sentences using the following words at least once:

- function;
- graph of a function;
- domain and range of a function;
- linear function;
- relative position of two straight lines;

- quadratic function;
- power function;
- function of inverse proportionality.

1. The function is defined by the formula $y = 4x - 5$.

Complete the table:

x	–2	–1,5	0		
y				3	11

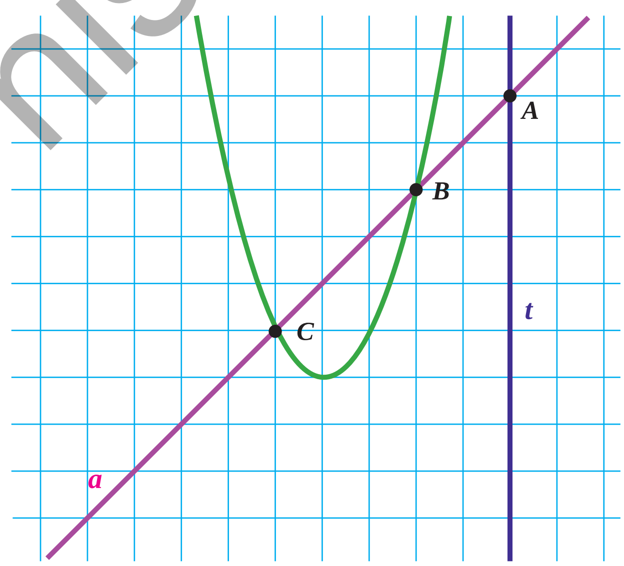
a) What is consistent with (–2)?
b) What is consistent with 2?
c) What is the value of a function, if the value of the argument is (–1,5)?
d) What is the value of an argument, if the value of the function is (–5)?

2. Define a function analytically, if:

a) the value of the function is 5 greater than the values of an argument;
b) the value of the function equals to the doubled value of an argument;
c) the value of the function equals to the tripled square of the value of an argument;

3. Arman plotted the graph of a function, but the coordinate axes have been erased:

a) Complete the graph with coordinate axes so that the designated line is a graph of the specified function.
b) Write the formulas for two other lines.
c) Specify the coordinates of points *A*, *B* and *C*. (a unit segment - one cell).
d) Will the line l cross $f(x)$ parabola? If yes, specify the coordinates of intersecting points.



4. Work with the function $y = -2x + 30$.

a) The graph of the function is … .
b) The graph of the function forms … an angle with a positive direction of *Ox* axis, because … .
c) The graph intersects the ordinate axis at (… ; …).
d) Draw a schematic graph of the given function.
e) Define a function by the formula the graph of which intersects the point (0; 0) in parallel with the given function.

5. Solve a system of equations graphically:

$$\begin{cases} 2x - y = 3, \\ 3x + 2y = -6. \end{cases}$$

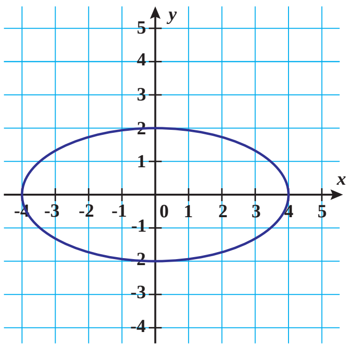
6. Find the values of a parameter b, at which the system

$$\begin{cases} 2x - by = 5, \\ 3x + 2y = 6 \end{cases}$$ **does not have solutions.**

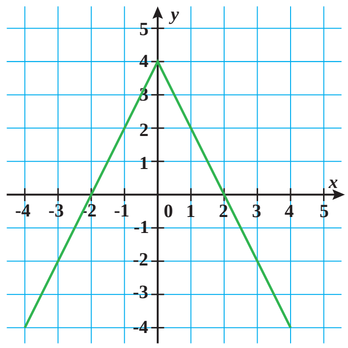
4.13 What do I know? Assessment activities

1. Which of the following lines represents a graph of a function? Why? Explain your answer.

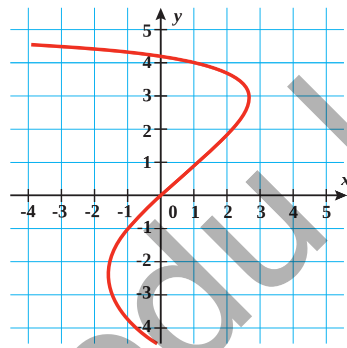
a)



b)



c)



2. Work with functions.

a) Plot the graph of the function $y=\frac{1}{3}x+2$.
b) Fill the gaps:

1. The graph of the function $y=\frac{1}{3}x+2$ intersects the ordinate at $A(\square;\square)$, and the abscissa at $B(\square;\square)$.

2. For the given function $y=\frac{1}{3}x+2$ it is possible to calculate values of the function with

the given values of an argument and vise versa, values of the argument with the given values of the function.

$f(-3)=\dots$; $\qquad f(60)=\dots$;

$f(x)=-1$, if $x=\dots$; $\qquad f(x)=23$, if $x=\dots$.

3. Points $M(8;\square)$ and $N\left(\square;\ -1\frac{1}{3}\right)$ belong to the graph of this function.

Highlight the part of the graph with a coloured pencil that includes the points with positive values of the abscissa and negative values of the ordinate.

3. Specify the functions satisfying the following conditions:

Line / Condition	$x+y=4$	$y=3x-2$	$x=4-y$	$2x+3y+1=0$	$y-1=2(x+2)$
The graph of the function intersects the point $(-2;\ 1)$					
The graph of the function is parallel to a line $y=-x$					
The graph of the function on coordinate axes intercepts congruent segments					
The graph of the function forms an acute angle with abscissa					

4. What is the value of parameter *k*, at which the graphs of the functions $y=(k+3)x-1$ and $y=(2k-1)x+3$ are parallel? Plot these graphs.

5. Plot the graph of the function $y=\begin{cases}-2x^2, & \text{if } x\le 0,\\ \dfrac{4}{x}, & \text{if } x>0.\end{cases}$

Find a domain of the function.

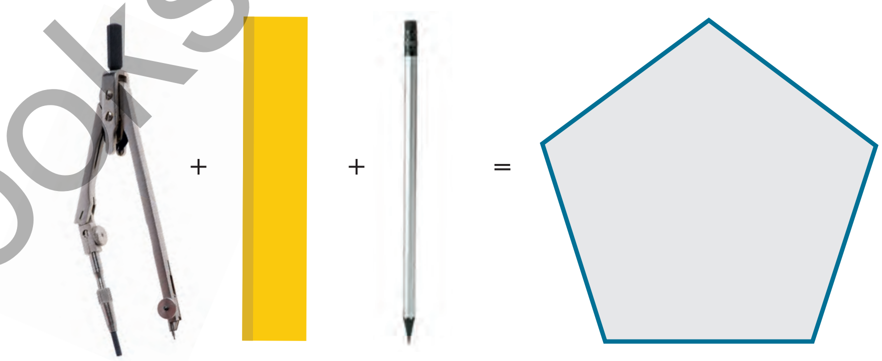
5 Circumference. Problem solving

By the end of this unit,
you will have learned:

✔ about circumference and circle, their similarities and differences; their position in the plane.
✔ circumference and a line, two circumferences;
✔ what are inscribed and circumscribed circumferences;
✔ what problems can be solved using a compass and a ruler

You will be able to:

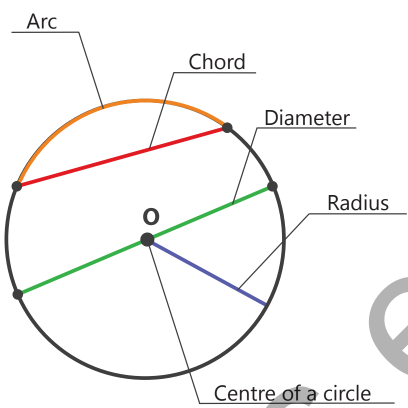
✔ determine relative position of a line and a circumference;
✔ determine relative position of two circumferences;
✔ construct using a compass and a protractor.

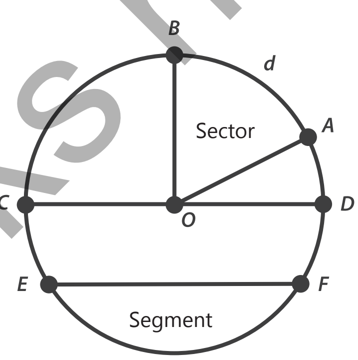


5.1 Circle and disk

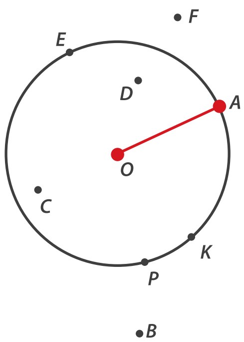
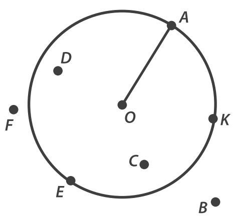
There are many different lines you can draw, but one that stands out of them is circle. In this lesson we are going to learn more about it.

<table>
<tr><th>Circle</th><th></th><th>Disk</th></tr>
<tr>
<td>Circle — is a set of all points in the plane equidistant from a given point. This point is called the centre of a circle.</td>
<td>Circle

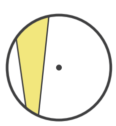
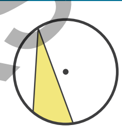
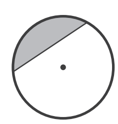
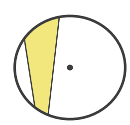

</td>
<td>Disk — is a set of all points in the plane bounded by a circle.</td>
</tr>
<tr>
<td>Radius of a circle — any segment connecting the centre of the circle with a point on the circle. The radius of the circle is labelled with R. A circle with a given centre O and a radius R is labelled as follows: $\omega(O;\ R)$. A circle divides the plane into interior and exterior regions.
Chord — is a segment that passes through two points of the circle.
Diameter — is a chord that passes through the centre. Chord is labelled with D.
Ark — is a part of a circle between its two points. Ark is labelled with a symbol —«◡».</td>
<td>Disk


</td>
<td>Centre of a disk — is the centre of circle.
Radius of a disk — is the radius of ifs circle. Radius of a disk is labelled with R.</td>
</tr>
</table>

1. Work with the drawing and fill in the gaps. Use definitions of a circle and a disk.

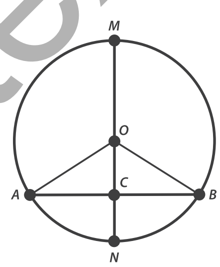
	a) The drawing illustrates a geometric figure with the centre at the point … and the radius R equal to the segment … . b) The segments … can form the radii of the circle. c) Points … lie on a circle. d) The points … do not belong to the circle. e) The points … lie in the interior region, and the points … lie in the exterior region of the circle. f) If we connect the centre of the circle … with the points …, then the lengths of the segments … will be less than R. g) If we connect the centre of the circle … with the points …, then the lengths of the segments … will be greater than R. h) From point E we can draw the chords … .
	a) The drawing illustrates a figure … with the centre at a point … and a radius R …. b) We can draw radii of the disk by connecting the points …. c) The points … do not belong to the disk. d) From the point A you can draw chords ….

2. Match between the definition of a figure and its drawing.

Sector is a part of a disk bounded by its two radii.			**A segment** is a part of a circle bounded by its arc and chord.	
 A	 B	 C	 D	 E

3. Прокомментируй доказательство утверждения приведенного ниже. Верно ли оно? Поясни свой ответ.

Theorem. If the diameter of the circle passes through the middle of the chord, then the diameter is perpendicular to the chord.

	**Given:** $\omega(O; R)$, AB — chord, C — midpoint of chord AB, MN — diameter. **Prove:** $MN \perp AB$. **Proof:**
Construct segments AO and OB	Connect the segments O and A, O and B.

ΔAOB — isosceles with the base AB.	Since sides OA and OB of the triangle are equal as radii.
OC is a median.	Since ΔAOB is isosceles.
$OC \perp AB$	OC is an altitude by the property of a median of an isosceles triangle drawn to the base.
$MN \perp AB$	Since the segments MN and OS lie on the same line, which means that if $OC \perp AB$, then $MN \perp AB$.
Which proves the theorem.	

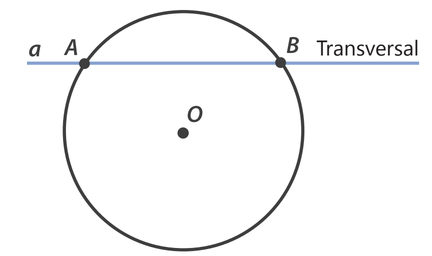
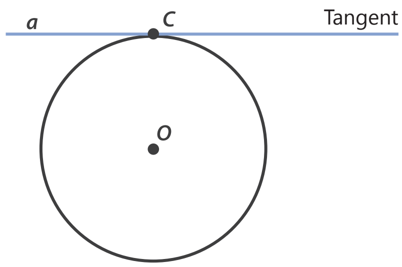
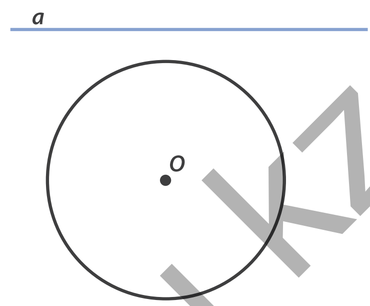
Will the converse theorem be correct? If so, then try to formulate and prove it.

4. Given a circle with a centre O, the diameter *MK* is perpendicular to the chord *AB*, which is equal to the radius. Find the angle *AOB* if *OA* = 12 mm.

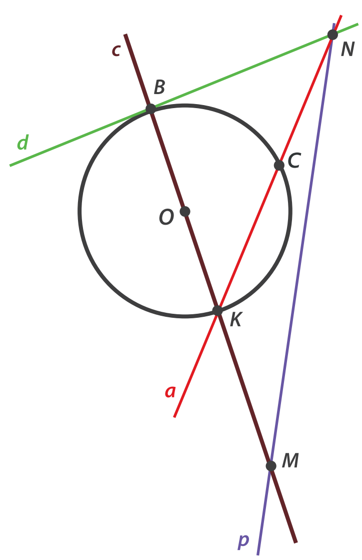
5. Given a circle with a centre *O*. The diameter *TC* intersects the chord *MK* in its midpoint — the point *P*. Angle *MOK* = 120°. Find the distance between points *T* and *P* if the chord is at the distance of 11 cm from the centre of the circle.

5.2 Relative position of a line and a circle

1. Describe the relative positions of a line and a circle.

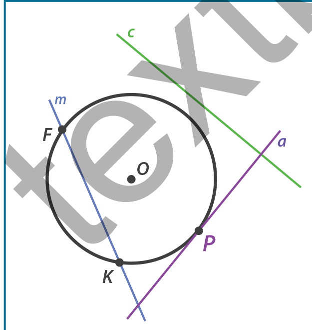
		
Line a is called a transversal if it has two common points with a circle $\omega(O;r)\cap a = A$, $\omega(O;r)\cap a = B$,	Line a is called a tangent is it has one common point with a circle. The common point is called tangency point. $\omega(O;r)\cap a = C$	

2. Work with a drawing and fill in the gaps:



a) The centre of the circle is a point … with a radius represented by a segment ….
b) The line … does not have common points with the circle.
c) The line d is … to the circle is and a point B is called … .
d) The lines … are transversal to the circle. The transversal … intersects the circle at the points … and …., and the transversal … intersects the circle at the points … and ….. The transversal … and … intersect at the point ….

3. Investigate.

	1. Measure the radius of the circle. 2. Find the distance from the centre of the circle to the lines a, c and m. 3. Compare the radius of the circle and the distance from the centre of the circle to each line.

REMEMBER!

A tangent to a circle is perpendicular to its radius drawn to the tangency point.

Fill in the table and draw a conclusion:

Line	Radius of a circle r	Distance d from the centre of the circle to the line	Comparison of circle radius and distance	Relative position of a line and a circle
a	$OP = ...$	$OP =...$	$r = d$	are tangent
m				
c				

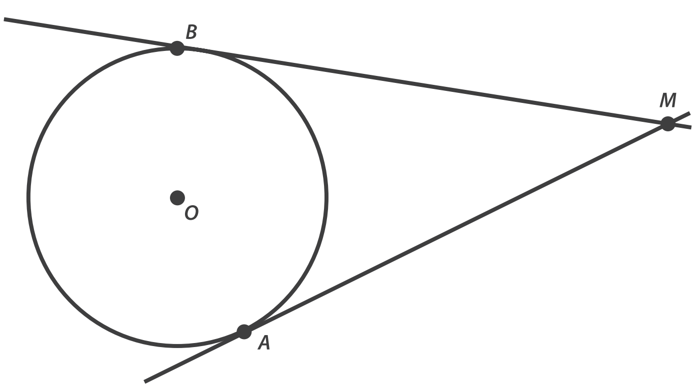
4. Given a circle $\omega(O; R=40$ мм). Use given data table to determine relative position of a line and the circle.

Line	The distance from the centre of the circle to the line	Position of the line and the circle	The number of common points
m	3 cm		
n	0,4 dm		
p	0,4 m		
k	0,07 m		
f	$\frac{1}{25}$ m		

5. Complete the drawing following the plan. Construct using a compass, a ruler and a set square.

1. Draw a circle.
2. Draw three parallel lines so that one of them is tangent to the circle, the second line is a transversal to the circle, and the third does not intersect the circle.
2. Label the lines, centre of the circle, intersection points.
3. Use a mathematical language to describe a position of each line relative to the circle.

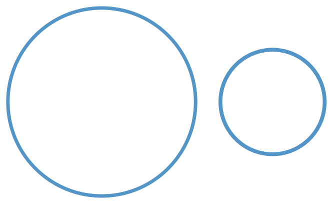
6. Marat drew a circle. From the point M lying outside the given circle, he drew two tangents to it. Marat believes that the distances from the point M to the tangency points are equal. Is he correct?



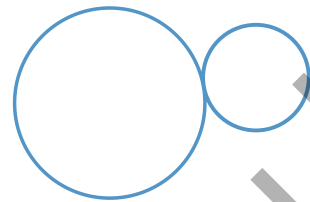
5.3 Relative position of two circles

1. Aliya drew two circles in various relative positions. Comment her drawings. Did she drew all possible relative positions?

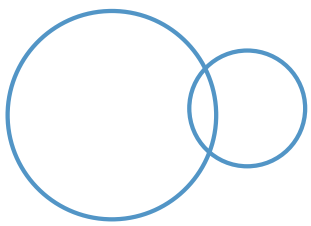
a)



b)



c)

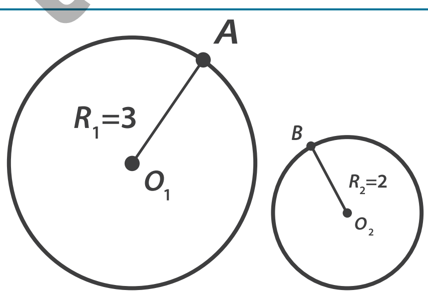


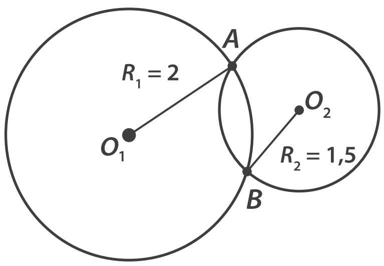
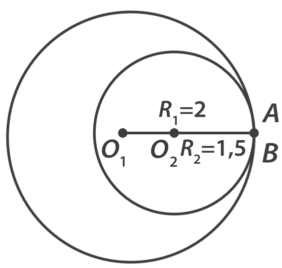
d)

REMEMBER!

- Circles with one common point are called tangent.
- If the centre of one circle lies in the exterior region of the other and the circles have one common point, then they touch externally.
- If the centre of one circle lies in the interior region of the other and the circles have one common point, then they touch internally.
- Circles are called concentric if their centres coincide.
- If circles have two common points, then they are called intersecting.

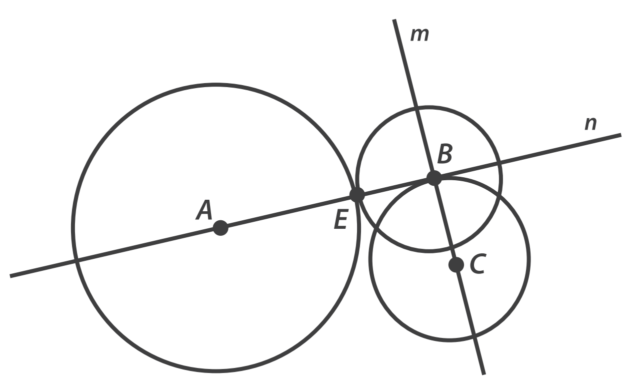
2. Fill in the table. Comment your answer.

Drawing	R_1	R_2	Compare (R_1+R_2) or (R_1-R_2) c O_1O_2	Conclusion
	3	2	$O_1O_2 > R_1+R_2$	Two circles do not common points if the distance between their centres is greater the the sum of their radii.

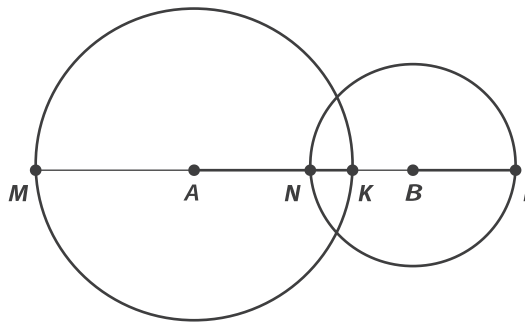
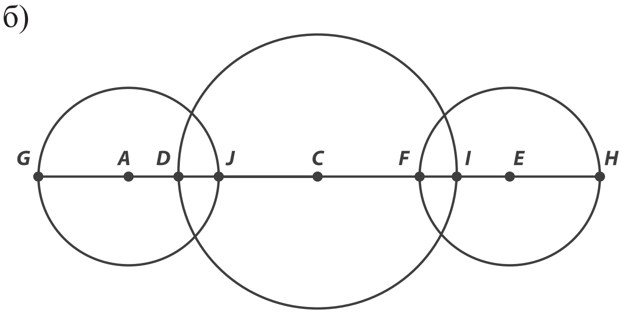
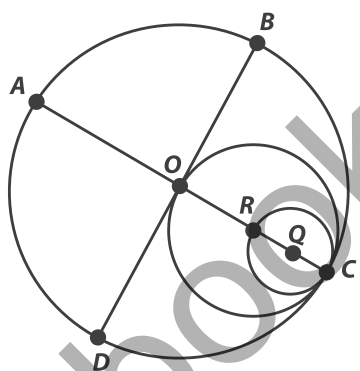
				Two circles do not common points if the distance between their centres is less the the difference of their radii.
				
			$O_1O_2 > R_1 - R_2$	
				The circles touch externally is the distance between their centres equals the sum of their radii.
				

3. Work with a drawing and answer the questions:

a) Which of the circles are tangent? In what way?
b) Which of the circles do not intersect?
c) Which of the circles intersect?
d) Which line is tangent and to which circle?
e) Which line is transversal and to which circle?
f) Which line and circle do not intersect?
g) Which line has common points with three circles?



4. Use ready made drawings to solve the problems:

а) 	**Given:** $\omega(A; 7)$, $\omega(B; 5)$, $NK = 1$. **Find:** AN, MP.
б) 	**Given:** $\omega(A; 12)$, $\omega(C; 32)$, $\omega(E; 12)$, $AD=IE=3$. **Find:** DJ, CD, CF, FI, AE.
в) 	**Given:** OB=16 см. **Find:** RQ, OQ.

5. Each of three circles touch two others.

a) How may the centres of the circles be positioned? Consider all possible options.

b) Calculate the perimeter of a triangle formed by the centres of these circles if their radii are 7 cm, 20 mm and 0.9 dm.

c) The radii of the circles are related as 1: 2: 3. Find the lengths of the radii if the perimeter of the triangle formed by the centres of the circles is 96 dm.

c) The centres of the circles form an equilateral triangle with a perimeter of 60. Find the radii of the circles.

d) The centres of the circles lie on the same line. Find the radius of the larger circle if radii of the other two circles of 15 cm and 19 cm.

6. The Earth and Mars move around the Sun in circular orbits with radii of 150 and 228 million kilometres respectively. What is the largest and smallest distance between Earth and Mars?

5.4 Locus of points

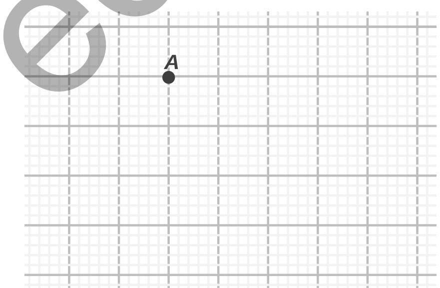
You already know that any geometric figure is a set of points on the plane. You can mark these points both arbitrarily and in a certain order. In geometry we pay more attention to the the points that have particular properties. In this case, such figures are called a geometric locus of points (GLP).

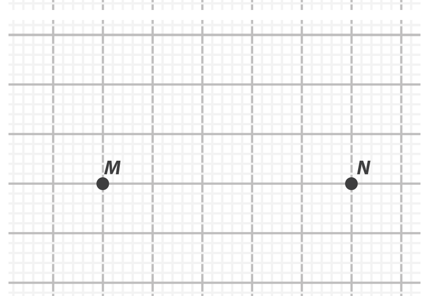
The locus of points (LOP) is a set of points in a plane that has a certain property. Moreover:
a) If a point belongs to a figure, then it has this property.
b) If a point has this property, then it belongs to the figure.

The best known locus of points for you is a circle, because it is a set of points in the plane that are at a certain distance from one point (which one?).

1. Mark the points in the grid nodes that are at a distance of:

a) 3 blocks from a point A;
b) less than 3 blocks from a point A;
c) less than 5 blocks, but more than 3 blocks from a point A;
d) less than 4 blocks from a point M and less than 3 blocks from a point N;
f) more than 3 blocks from a point M and less than 2 blocks from a point N.



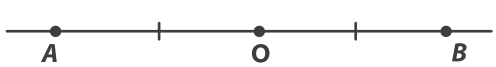


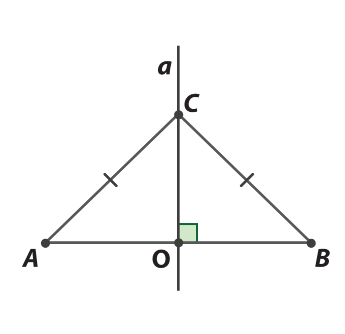
2. Construct following the plan and answer the questions:

1. Draw a segment AB of 5 cm long.
2. Show the locus of points that are at a distance of 4 cm from point A.
3. Show the locus of points that are at a distance of 2 cm from point B.
4. Find the set of points that are at a distance of 4 cm from a point A and 2 cm from a point B.
5. Find the set of points that are at a distance of less than 5 cm from a point A and more than 4 cm from a point B.

3. What is the locus of points in plane that are at a distance of 3 cm from a line *a*. Explain your answer.

4. What is the locus of points in the plane that are equidistant from the ends of a given line segment?

	**Given:** AB — segment. **Find:** LOP equidistant from A and B.



Solution:

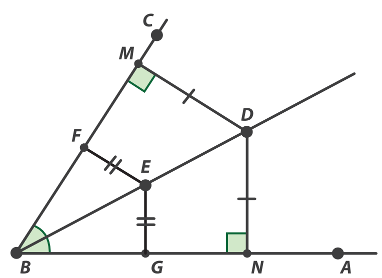
Assume that point C is equidistant from points A and B of the segment. Then ΔABC is isosceles (why?) and the point C lies on a line containing the median, bisector and altitude of the given triangle. In relation to the segment AB, this line will be a perpendicular passing through the midpoint of the segment AB. Therefore, the unknown LOP is a perpendicular that passes through the midpoint of the segment AB.

5. Is it true that if the point belongs to the perpendicular bisector drawn to this segment, then it is equidistant from its endpoints? Explain your answer.

6. Describe the path formed by all points in the interior of an angle the same distance from both sides of an angle.

REMEMBER!

The perpendicular bisector to a segment is a line perpendicular to a given segment that passes through its midpoint.

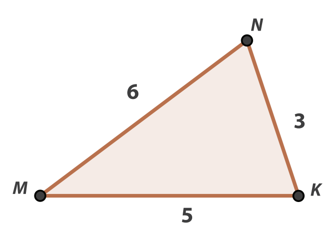


Given: ABC — angle.
Find: LOP equidistant from A and B.

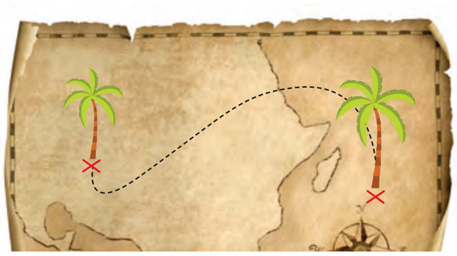
Solution:
1) Mark equal segments $BK = BP$ on the angle CBA, so that point D belongs to the segment KP. We will get an isosceles triangle KBP. The angle of BKP will be equal to the angle BPK since the angles at the base of the isosceles triangle are equal.
2) Triangles KMD and NDP are rectangular. The angles MDK and NDP are equal, as the difference of the angle 90^0 and equal angles MKD and NPD. $MD = ND$ as given.
The triangles KMD and NDP are congruent by the second condition, and therefore $KD = DP$. Therefore, the point D is the midpoint of the segment KP.
3) Construct a half-line BD. In the isosceles triangle BKP the segment BD is a median and a bisector of the angle of the KBP. Therefore, BD is the bisector of the angle ABC. The unknown LOP is the bisector of this angle.

7. Is it true that if the point lies on the angle bisector, then it is equidistant from its sides? Explain your answer.

8. Ruslan drew a triangle MNK. What locus of points is the point N? Explain your answer.



9. Pirate Joe hid a treasure on Treasure Island. He remembered that the treasure is located near two palm trees, which are located at a distance of 15 feet from each other. He marked those palms on the map. The treasure is 12 feet from the first palm, and 10 feet from the second. Find possible locations of the treasure.



5.5 Construction with compass and ruler

As you already know, the main tools to construct geometric figures are a ruler and a compass. We can use them to construct a figure with desired properties. But there are a number of rules that should be followed. All constructions should be performed only by a compass and a ruler without scales.

Now we are going to consider what constructions we can perform by a ruler, and which by a compass.

Use a ruler to prove:	**Use a compass to prove:**
• part of any line; • part of a line passing through a given point; • part of the line passing through two given points.	• a circle of a given radius centred at a given point; • mark off a given segment from a given point.

A compass and a ruler are typically used in constructing geometric figures, but there are a number of problems that can be solved only with the help of a compass or only with a help of a ruler.

1. What geometric figures we construct using:
a) compass;
b) ruler?

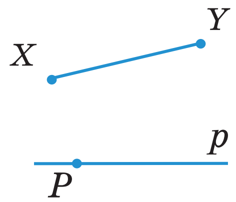
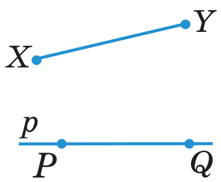
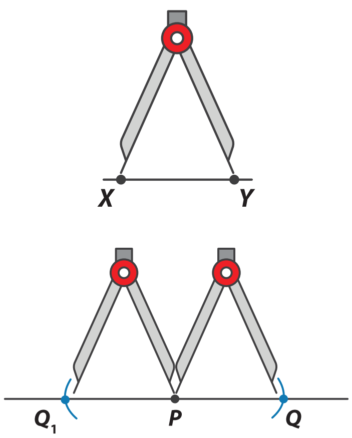
All problems on constructing a figure with given properties are called construction problems.

To solve a construction problem is to find a way to construct a figure, construct it and prove that the constructed figure has the given properties.

Challenge to build	
Step 1 - Analysis Analysis of the source data and construction plan making.	**Step 2 - Construction** Implementation of construction according to the plan
Step 3 - Proof Prove that the constructed figures satisfy the condition of the problem	**Step 4 - inquiry** It is necessary to answer the questions: Does a task always have a solution? If so, how many?

Sometimes, when a problem is quite simple, some steps of the solution can be skipped if they are obvious.

2. Look through the solution and and give your comments.

	Mark off a line p equal to the given segment XY from the given point P. **Solution:**
	**Analysis.** Assume that the problem is solved and the desired segment PQ is constructed. Since $PQ = XY$, the point Q lies on a circle centred at the point P with a radius XY.
	**Construction.** 1. Construct the circle $\omega(P; XY)$; 2. $\omega(P; XY) \cap p = Q$, $\omega(P; XY) \cap p = Q_1$; 3. PQ, PQ_1 — desired segments. **Proof:** $PQ = PQ_1$, as they are radii of the same circle and by the axiom of the segment measuring. **Inquiry.** The problem has two solutions: PQ, PQ_1.

3. Given points M and N. Use a compass to construct a point K so that $MK = 3MN$.

4. Given segments of x and y cm long. Construct the segments that are equal to:

a) $x + y$;
b) $x - y$;
c) $2x + y$;
d) $2x - y$.

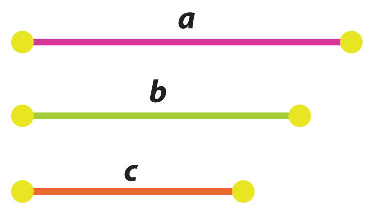
In which case the problem will not have a solution? Explain your answer.

5. Construct two equal chords of a circle, emerging from one point.

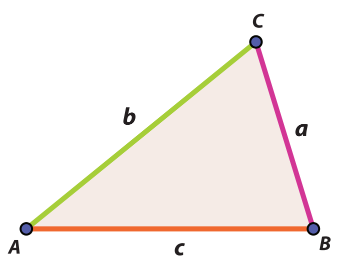
5.6 Constructing a triangle when three sides are given

You already know how to construct a segment that is equal to a given one. This will help you to construct geometric figures, including a triangle, when the sides are given.

1. Construct a trinagle *ABC*, if its three sides *a*, *b*, *c* are given.

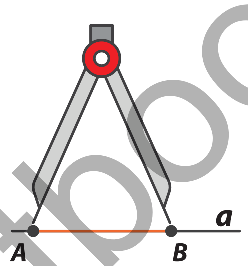


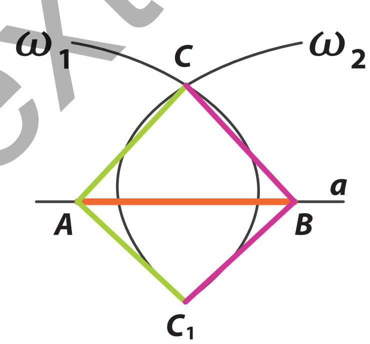
Given: *a, b and c — and sides of a triangle ABC.*
Prove: ΔABC .



Analysis.
Assume the problem is solved, given triangle *ABC* is constructed and $AB = c,\ AC = b,\ BC = a$.

Since the vertex *C* is at a distance of *a* and *b* from the vertices *A* and *B*, it is the intersection of two locus of points:
$C = \omega_1 (A;\ R = b) \cap \omega_2\ (B;\ R = a)$.





Construction.
1. Draw an any line a, mark a point *A*.
2. Mark off a segment *AB* equal to the segment *c* from a point *A*.
3. Draw two circles:
   $\omega_1 (A;\ R = b)$ and $\omega_2\ (B;\ R = a)$.
4. Label one of the intersection points of these circles with a letter *C*, the second with a letter C_1.
5. Construct segments *AC*, *BC* and AC_1, $BC_{1.}$
   ΔABC and ΔABC_1 unknown.

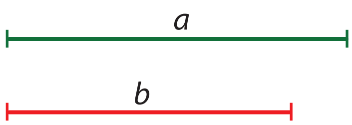
Proof.

$AB = c$ by construction,
$AC = AC_1 = b$, since AC — is a radius of a circle ω_1,
$BC = BC_1 = a$, since BC — is a radius of a circle ω_2.
That means the constructed triangle has sides that are equal to the given segments.

Inquiry.

The problem has no solution if at least one of the triangle inequalities is violated:
$\grave{a} < b + c,\ b < a + c,\ c < a + b$.

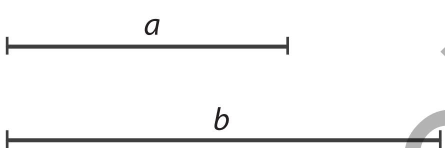
2. Construct an isosceles triangle *MNK* with sides equal to line segments *a* and *b*.



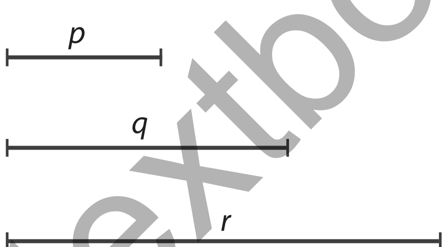
3. Construct an equilateral triangle with a side equal to a segment *m*.

4. Is it possible to construct a triangle *ABC* if:

a) it is an isosceles triangle and its lateral sides are equal to the segments *a* and *b*:



b) lateral sides of the triangle are equal to the segments *p*, *q* and *r*:

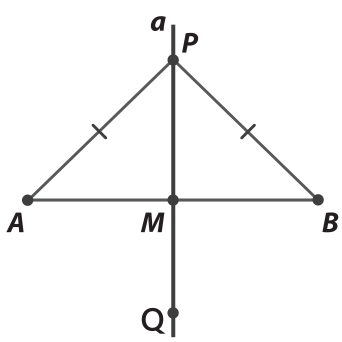


5. Construct a triangle *ABC* if given two sides *AB* and *AC*, and a median *CD*.

5.7 Constructing a perpendicular bisector. Constructing a perpendicular to a line

1. Construct a perpendicular bisector to a given segment *AB*.

1. Work with the drawing and analyse the construction of a perpendicular bisector to a given segment *AB*.

	**Given:** A segment *AB*. **Построить:** A perpendicular bisector to *AB*.

2. Construct following the plan. Make the appropriate math notes.

a) Construct two circles with radius *AB* and centres at the points *A* and *B*.
b) They intersect at two points - *P* and *Q*.
c) Draw a line *PQ*.

This line is the unknown perpendicular bisector drawn to the segment *AB*.

3. Proof. Prove and fill in the gaps.

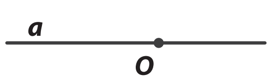
a) By the construction, points … and … are equidistant from the endpoints of the segment *AB*, so they lie on … to this segment, based on the theorem of ….
b) Therefore, … passes through the points … and … to the segment *AB*, which means, it superimpose with the line ….

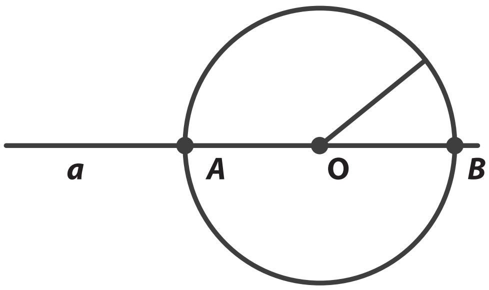
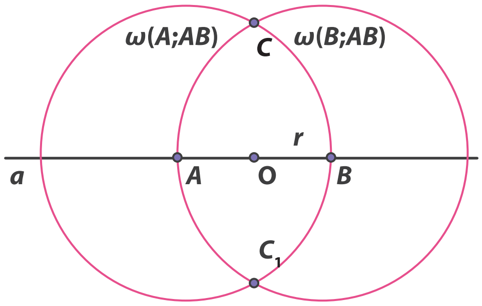
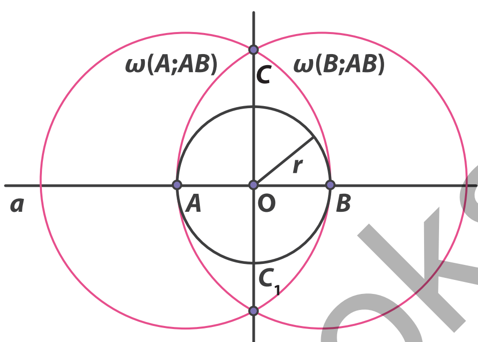
4. Inquiry. How many solutions does the problem have? Explain your answer.

2. Complete and comment the solution for a construction problem:
Construct a line that passes through a given point *O* and is perpendicular to a given line *a*.

There are two possible options: when the point *O* belongs to a given line, and when it does not.

Case 1.

	**Given:** *O* belongs to *a*. **Prove:** Line *b*: $b \perp a,\ O \in b$.

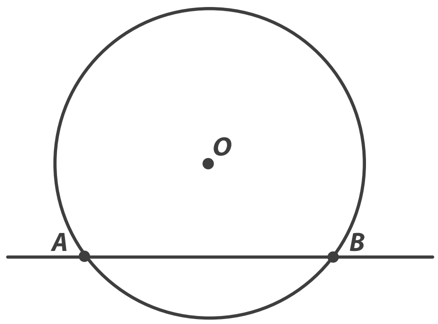
	1. Draw a circle $\omega(O; r)$, where r — is any radius: $\omega(O;r)\cap a = A$, $\omega(O;r)\cap a = B$
	2. Draw circles $\omega(A;\, AB)$, $\omega(B;\, AB)$: $\omega(A;\, AB)\cap\omega(B;\, AB) = C$, $\omega(A;\, AB)\cap\omega(B;\, AB) = C_1$
	3. Draw a line OS that passes through points O and C. The line OS is the unknown line b.

Case 2.

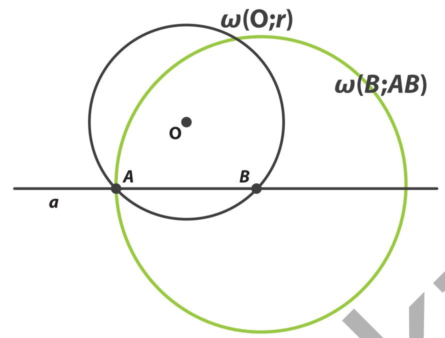
	**Given:** $\boldsymbol{a}$ — is a line. **Prove: A line** b: $b \perp a,\ O \notin b$.

Work with the drawings and write down the steps of a construction.

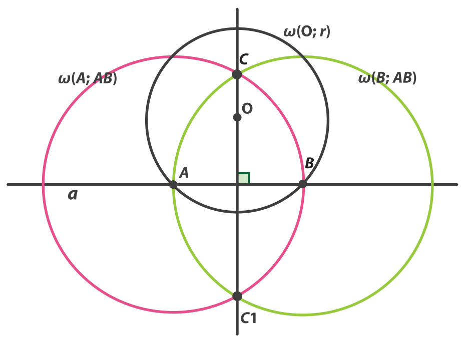
1.



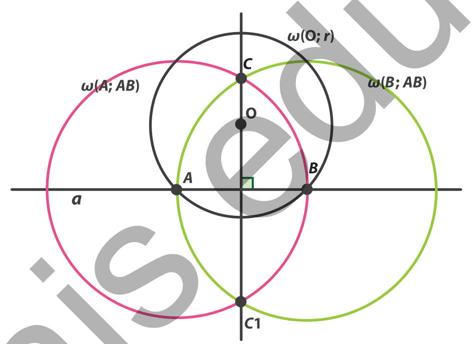
2.



3.



4.



3. Using the results of the previous problems, find a method how to divide a segment in half.

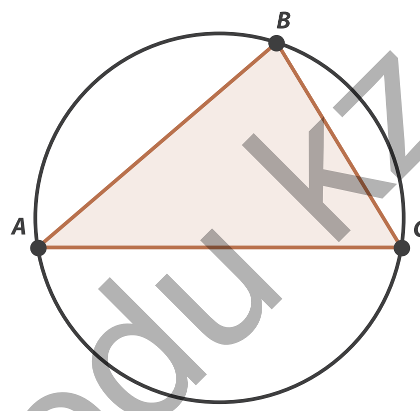
4. Use a compass and a ruler to draw all the medians of a triangle *ABC* with the sides *a, b,* and *c*.

5.8 Circumcircle of a triangle

A circle and a triangle are the two main figures of geometry. Consider how these figures can be positioned relative to each other.

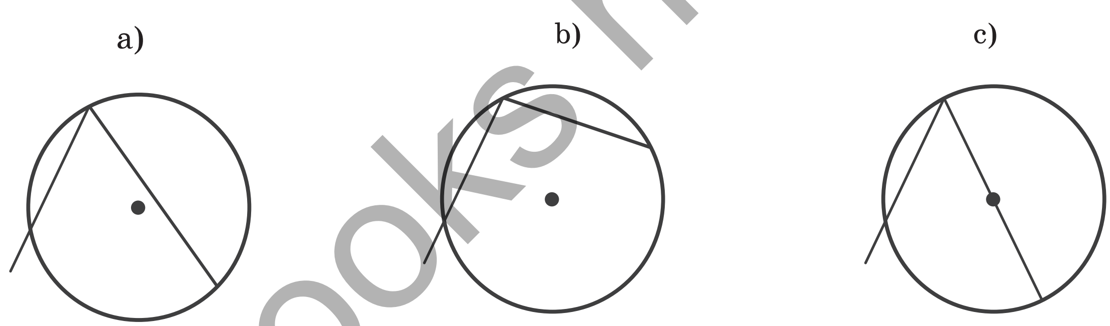
1. Draw a circle and a triangle so that they have 1, 2, 3, 4, 5, and 6 common points.

Consider the case when all three vertices of a triangle lie on a circle. In this case, it is called circumcircle.



A circle which passes through all three vertices of a triangle is called circumcircle of a triangle.

2. Complete the drawings to get circumcircle of the triangles.

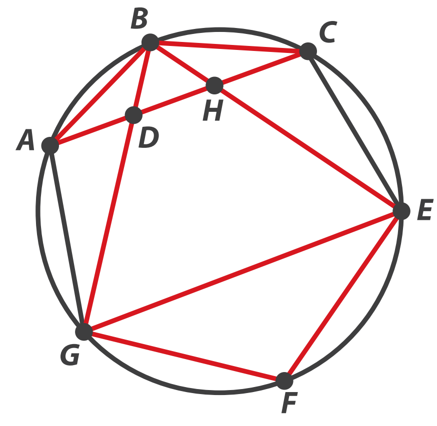


How can I determine the type of each triangle. How is the centre of the circle positioned relative to the triangle?

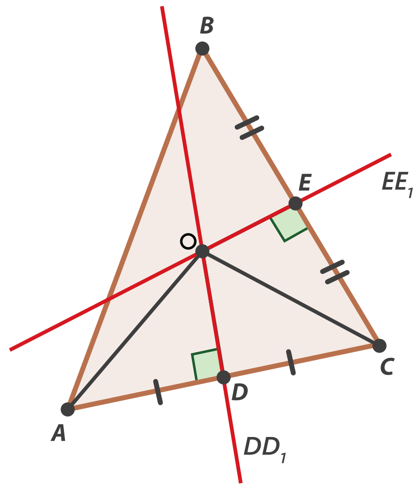
3. Work with the drawing and find all the triangles:

a) with circumscribed circle;

b) that are not inscribed in the circle.



4. Comment the proof of the theorem: Any triangle can have a circumcircle. The centre of this circle lies in the intersection of perpendicular bisectors.

	**Given:** ΔABC. **Prove:** O — intersection of the perpendicular bisectors.
Solution: Use the locus of points to find a point equidistant from points A, B and C.	
1. Draw $D_1D \perp AC$	D_1D — perpendicular bisector
2. Draw $E_1E \perp BC$	E_1E — perpendicular bisector
The perpendicular bisectors intersect at one point, which means $OA = OC$, $OC = OB$	$DD_1 \cap EE_1 = O$
3. $OA = OC = OB$	By item 3
4. Therefore the point O is the centre of the circumcircle, and the segments OA, OC, OB are the radii of this circle. Which proves the theorem.	

5. Use a compass and a ruler to construct:

1. Draw any triangle MNK and circumscribe of this triangle.
2. Inscribe two triangles into one circle so that they have a common side.

6. Prove that if:

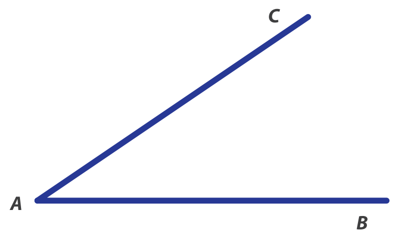
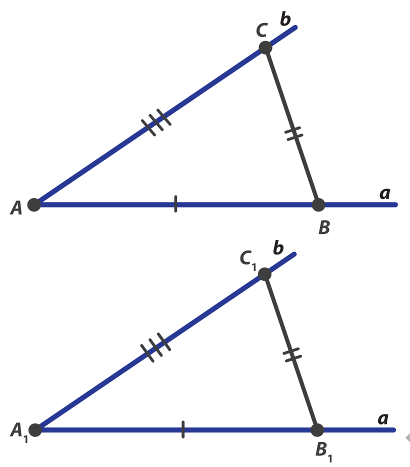
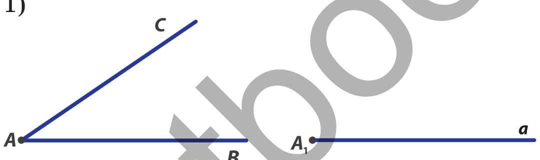
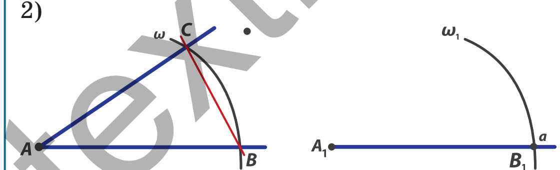
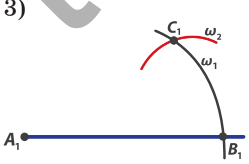
a) a centre of the circumcircle lies at the intersection of bisectors of the triangle, then this triangle is equilateral;

b) a centre of the circumcircle lies on the side of the triangle, then the given triangle is rectangular.

5.9 Constructing an angle equal to a given one. Constructing an angle bisector

In this lesson we continue using a compass and a ruler to construct an angle equal to a given.

1. Build an angle equal to the given one so that one of the sides coincides with the given half-line. Work with the drawing and analyse the construction:

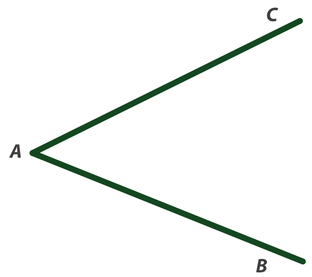
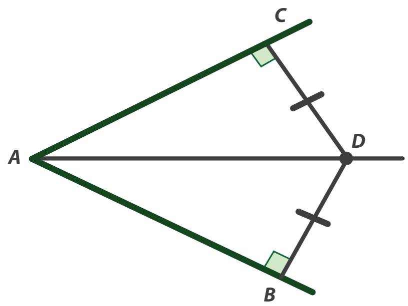
	**Given:** $\angle CAB$. **Prove:** $\angle C_1A_1B_1 = \angle CAB$.
	**Analysis.** Assume that the problem is solved, and the unknown $\angle C_1A_1B_1$ is constructed. Equal angles lie against equal sides in congruent triangles . It is enough to construct two congruent triangles with vertices at points C_1 and C to solve this problem.
Construct following the plan:	
1)  2)  3) 	**Construction.** 1. Assume given an angle *CAB* and a half-line *a*. $A_1 \in a$ 2. $\omega(A; AB)$, $\omega_1(A_1; AB) \cap a = B_1$ $\omega_2(A_1; AC)$ BC. 3. $\omega_2(A_1, BC)$. 4. $\omega_2(A_1, BC) \cap \omega_1(A_1, R) = C_1$. 5. A_1C_1 , B_1C_1 6. $\triangle ABC = \triangle A_1B_1C_1$ 7. $\angle C_1A_1B_1 = \angle CAB$

Complete Proof and Inquiry stages independently.

2. Given two angles A and B ($\angle A > \angle B$). Construct angles that are equal to:

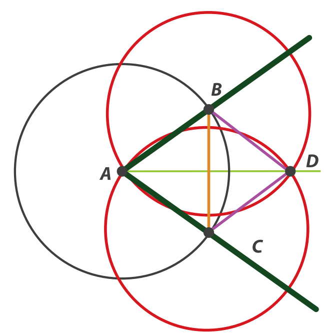
a) $2\angle A$; b) $\angle A + \angle B$; c) $2\angle A + \angle B$.

3. Construct a bisector of a given angle. Work with the drawing and analyse the construction:

	**Given:** $\angle CAB$. **Построить:** AD — is a bisector $\angle CAB$.
	**Analysis** Assume the problem is solved, and the bisector is constructed. To construct a bisector, it is necessary to construct a point D different from point A and equidistant from the sides of the angle (why?). Then the task comes down to constructing a triangle equal to the given one by two angles.

Construct following the drawing. Follow the steps below.

Construction.



1. Let the angle *CAB* be given.
2. Construct a circle with any radius r and a centre at a point *A*. Label the points of its intersection with the sides of the angle with letters *B* and *C*.
3. Construct two circles with the same radius *r* from the points *B* and *C*. They intersect at two points. Label the point of intersection that is lies on opposite to *A* side *BC* with a letter *B*.
4. Draw a half-line *AD*. This is the unknown bisector of a given angle *A*.
5. Проведем луч *AD*. Это and есть искомая биссектриса данного угла *A*.

Complete Proof and Inquiry stages independently.

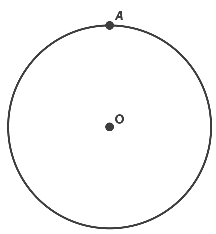
4. Construct an bisector of a right angle.

5. Half this angle.

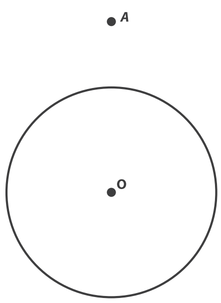
5.10 Construction of a tangent

1. Construct a tangent to a given circle passing through the given point.

Case 1.

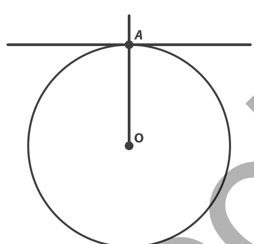


Case 2.



Given: $\omega(O, R)$,
Case 1: $A \in \omega$,
Case 2: $A \notin \omega$.

Построить: $\boldsymbol{a}$ — кis a tangent to a circle passing through a point A.



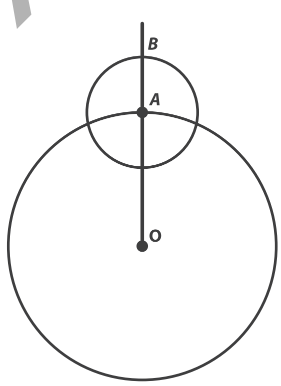
Analysis.

The task of constructing a tangent to a given circle comes down to determining a tangency point and constructing a line perpendicular to the radius drawn to the tangency point.

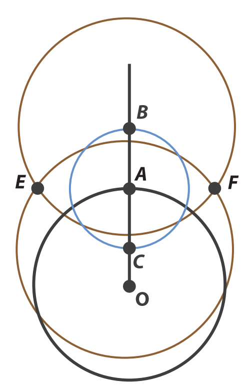
Construction.
Determine the correct construction order and make the appropriate notes.

Case 1.

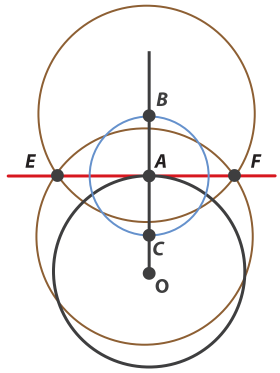
a)



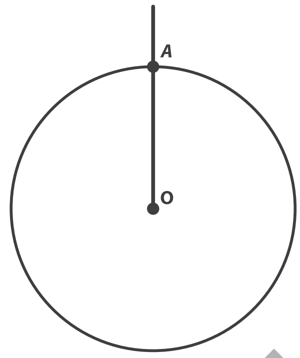
b)



c)

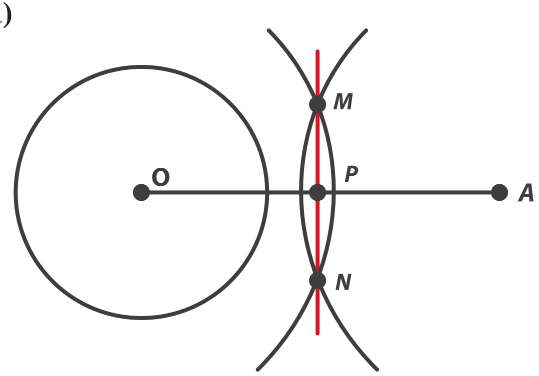


d)

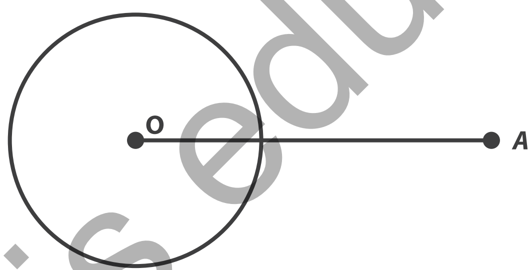


Case 2.

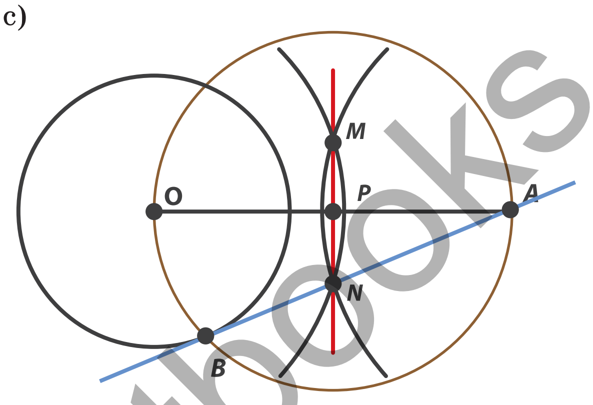
a)



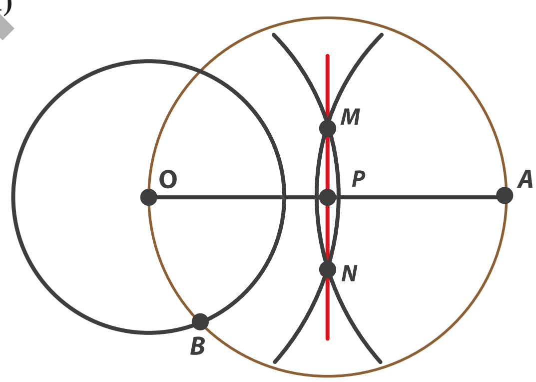
b)



c)



d)



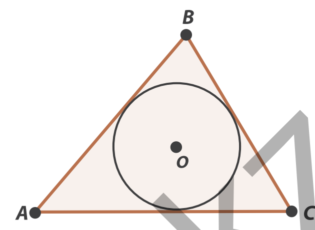
Complete proof and inquiry independently.

2. Given a circle and a line that does not have common points with the circle.
Construct a tangent to the circle:
a) parallel to the given line;
b) perpendicular to a given line.

5.11 Incircle of a triangle

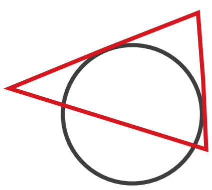
You already know that you can draw a circumcircle of a triangle. However a circle can also be inscribed in a triangle. In this lesson we will talk about this in more detail.

Incircle of a triangle is the circle that touch each side of the triangle.

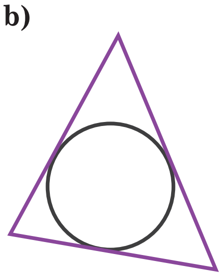


1. Which figure depicts a incircle of a triangle. Explain your answer.

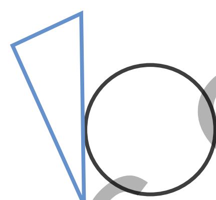
a)



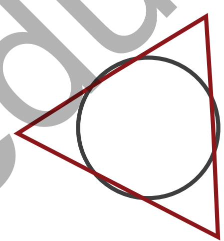
b)



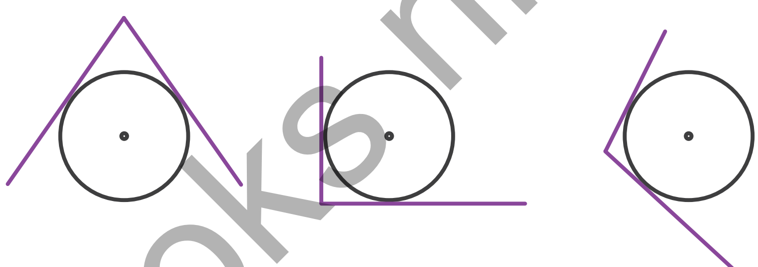
c)



d)

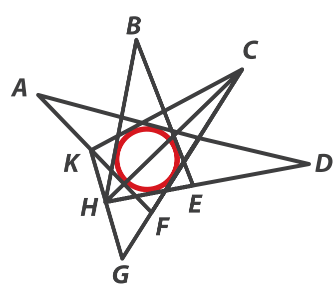


2. Complete the drawings to get incircle of the triangles.

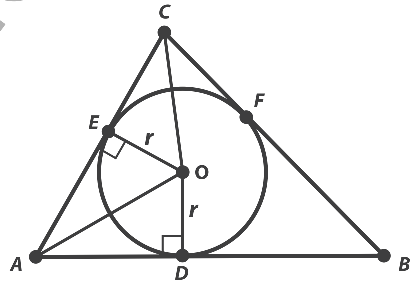


What type do the given triangles have? How does the centre of the circle is located?

3. Work with the drawing and find all triangles:
a) with the circle inscribed;
b) without the circle inscribed.



4. Comment on the prove of the theorem:
Any triangle can have an incircle. The centre of the given circumference is the intersection of angle bisectors.



Given: ΔABC.

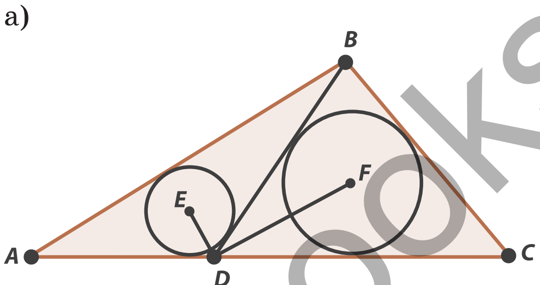
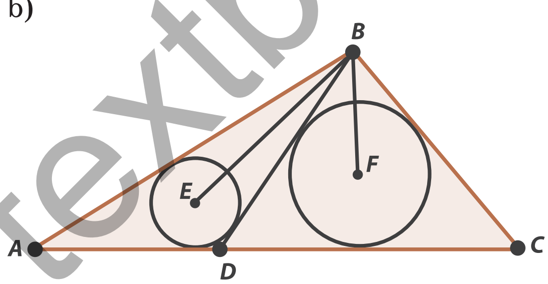
Prove: O — the intersection of bisectors of interior angles of the triangle.

Solution: Use locus of points to find a point equidistant from the sides of the triangle.	
1. Draw bisectors of the interior angles of the triangle.	They intersect at one point.
2. $OE = OD$	Since any point of the bisector of the angle is equidistant from its sides
3. $OF = OE$	Similarly
4. $OF = OD$	Similarly
5. $OE = OD = OF$	According to the points 2–4
6. Therefore, the point O is the centre of the incircle, and the segments OE, OF, OD are the radii of this circle. Which proves the theorem.	

5. Construct following the plan:

1. Draw any triangle.
2. Find the centre of the incircle of a given triangle.
3. Find the tangent points of the given circle with the sides of the triangle.
4. Construct an incircle of the triangle.

6. Solve the problems using ready made drawings.

a) 	**Given:** ΔABC, $\omega(E;\ r_1)$, $\omega(F;\ r_2)$ — incircles. **Find:** $\angle FDE$.
b) 	**Given:** ΔABC, $\omega(E;\ r_1)$, $\omega(F;\ r_2)$ — incircles, $\angle FBE = 35°$. **Find:** $\angle ABC$.

7. Prove that if

a) centres of an incircle and a circumcircle coincide, then the triangle is equilateral;
b) centre of the incircle lies at the altitude of a triangle, then the triangle is isosceles.

5.12 Constructing a right triangle

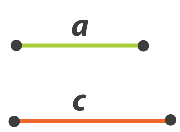
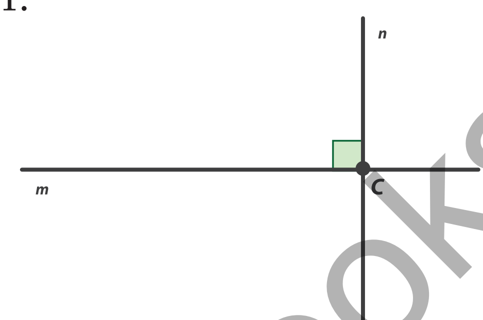
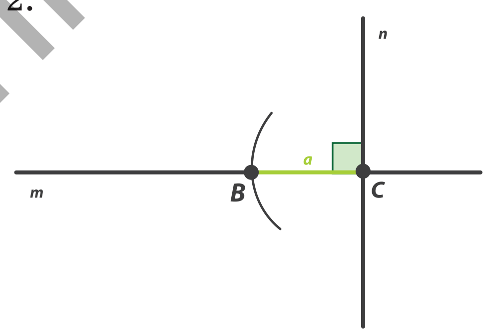
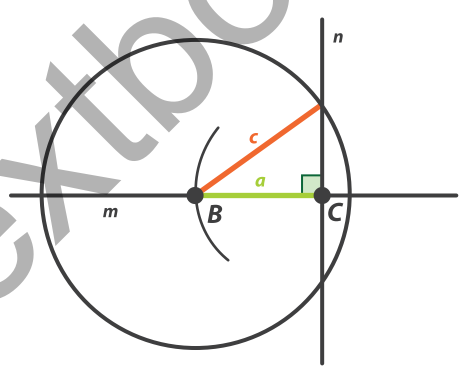
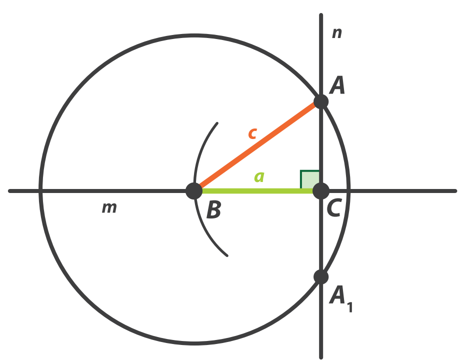
1. Use a compass and a ruler to construct a right triangle where one of its legs is 3 cm and the other is 4 cm.

2. Arman constructed a right triangle by a given hypotenuse and leg. He used two construction methods. Analyse his construction. Make appropriate notes.

REMEMBER!

The centre of the circumcircle of a right triangle lies in the midpoint of the hypotenuse

Method 1.

	**Given:** a — a leg, c — a hypotenuse. **Construct:** Right triangle ABC. $\angle C$=90, BC=a, AB=c.
Construction:	
1. 	2. 
3. 	4. 

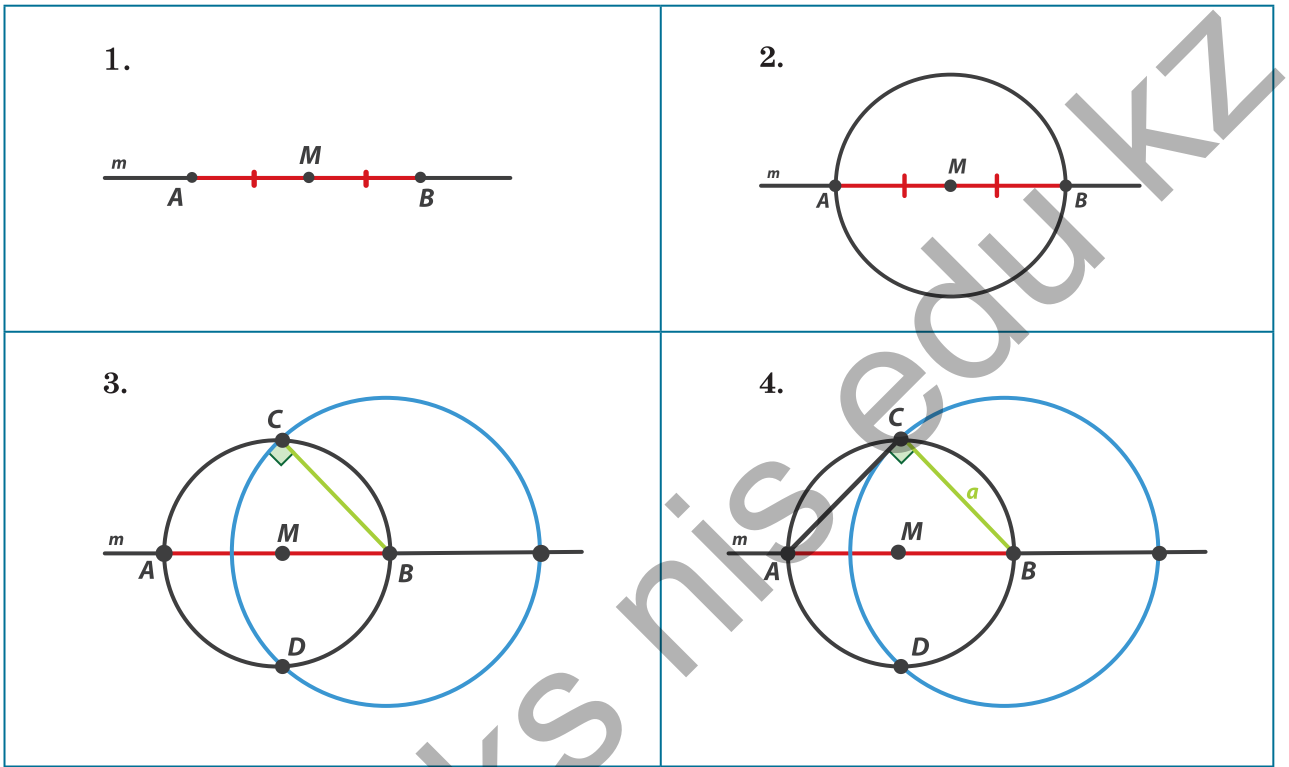
Proof:

Statement	Argumentation
$m \perp n$	By construction, that is C is a right angle;
$BC = a$	since BC is a radius of the circle $\omega_1\,(C; a)$;

$BA = c$	since BA is a radius of the circle $\omega_2(B;c)$.
Which means both triangles ABC and A_1BC — are unknown.	

Inquiry. Since the triangles ABC and A_1BC are congruent by a hypotenuse and a leg, the problem has only one solution.

Method 2.



A

Complete proof and inquiry independently.

3. Use a compass and a ruler to construct a right triangle:

a) with a given hypotenuse and acute angle;

b) with a given leg and the adjacent acute angle;

c) with a given leg and the opposite acute angle.

4. Is it possible to construct a right triangle:

a) with given two equal segments;

b) if one of its angles is 30°;

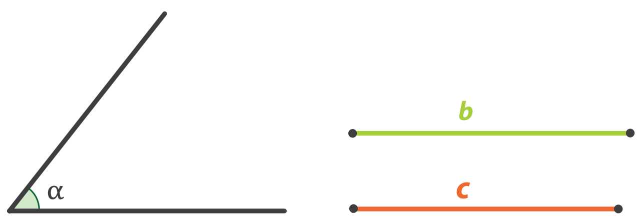
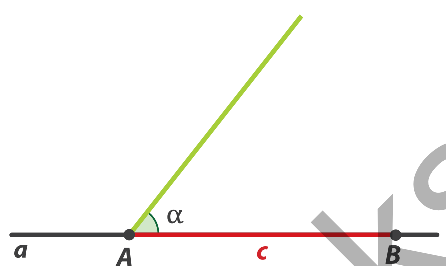
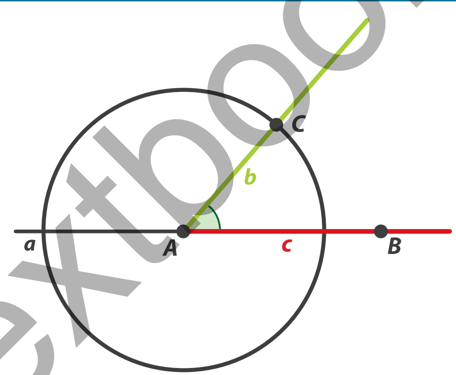
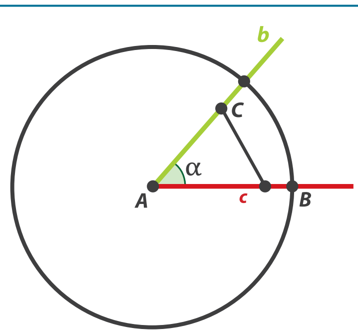
c) if one of its angles is 45°?

Construct the triangles.

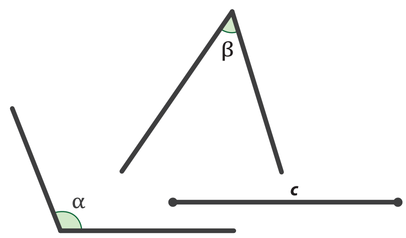
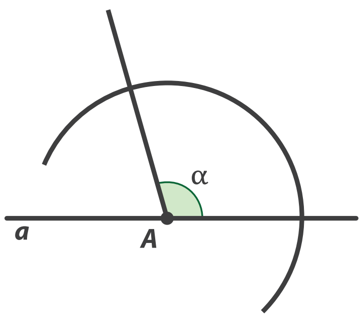
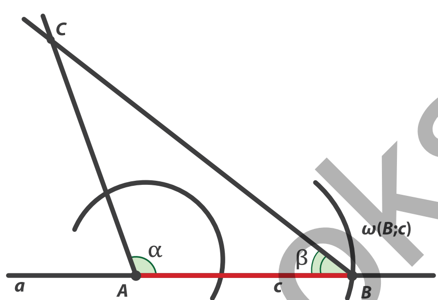
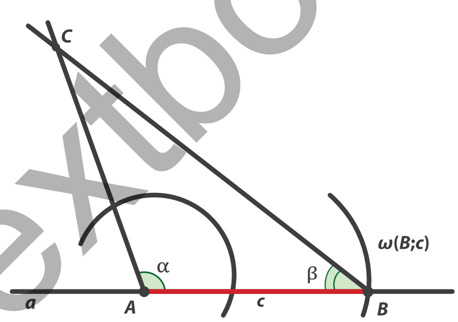
5.13 Constructing a triangle with given elements

You already know how to construct a triangle with three sides given. You also know how to construct an angle equal to the given one. On this lesson we will apply this knowledge to construct triangles.

1. Construct a triangle with the given elements and fill in the gaps:
a) two sides *b* and *c* and the angle between them *α*.

	**Given:** α — an angle of a triangle, a, b — sides of a triangle. **Construct:** ΔABC with the given elements.
	Mark a point A on the line … and mark off a segment … equal to ….
	Mark off an angle equal to … from a point A
	Construct a circle $\omega(\grave{A};\ R = b)$. This circle intersects the side of the triangle at a point C.
	Connect C and B with a segment segment. We will receive the unknown … ABC.

b) a given side c and two adjacent angles α and β.

	**Given:** α, β — angles of the triangle, c - a side of the triangle. **Построить:** ΔABC with the given elements.
	Mark a point A and mark off an segment … equal to … on the line … .
	Construct ... equal to a given angle α with a vertex at ... A.
	Construct an angle equal to a given angle β with a vertex at point B.
	Mark a point C - the intersection of half-line AC and BC. We will receive the unknown ... ABC.

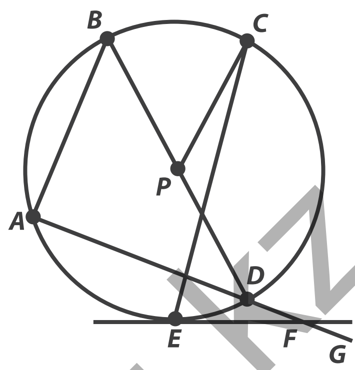
2. Construct an isosceles triangle.

a) with a given side and an angle at the vertex;
b) with a given base and an angle at the base;
c) with a given lateral side and an altitude drawn to the base.

5.14 Problem solving

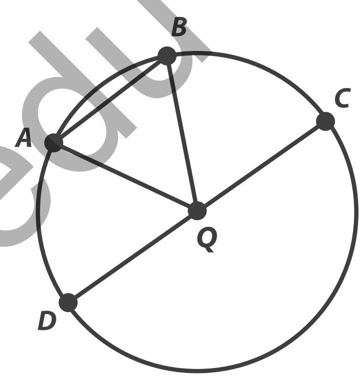
1. Work with the drawing and mark:

a) a centre of the circle;	b) a radius of the circle;
c) a diameter of the circle;	d) a transversal of the circle;
e) a tangent to the circle;	f) a triangle with circumcircle;
g) a triangle described about a circle;	h) points lying on a circle;
i) points that do not belong to a circle;	j) points lying inside a disk bounded by a circle.

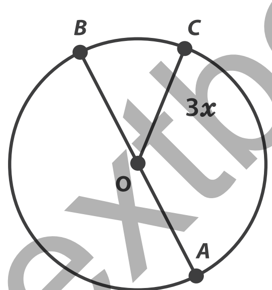
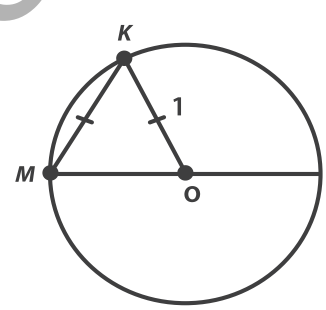
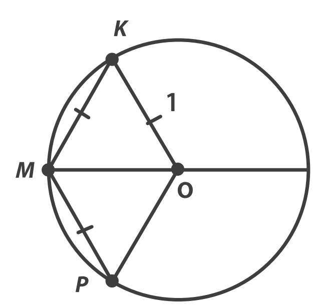


2. Given a circle ω with centre Q. Choose correct statements. Explain your answer.

a) $AQ+QB>AB$; b) $AQ=DQ$, $BQ=QC$;
c) $DQ+QC>AB$; d) $DC>AB$.



3. Solve problems using ready-made drawings:

a) Given: $AB=9x-24$ **Find:** OC. 	**b) Given:** $MK=MO=1$. **Find:** $\angle M$, $\angle O$, $\angle K$, $P(\triangle OMK)$ 	**c) Given:** $MK=MP=MO=1$. **Find:** $\angle M$, $\angle K$, $\angle P$, $P(MKOP)$ 

4. Given three circles ω_1 $(O, R=9$ cm$)$, ω_2 $(P, R=18$ cm$)$, ω_3 $(S, R=8$ cm$)$. Centres of the circles lie between two parallel lines. The distance between these lines is 18 cm. What circle:

a) has two transversals;
b) has two tangents;
c) does not have common points with each of the lines?

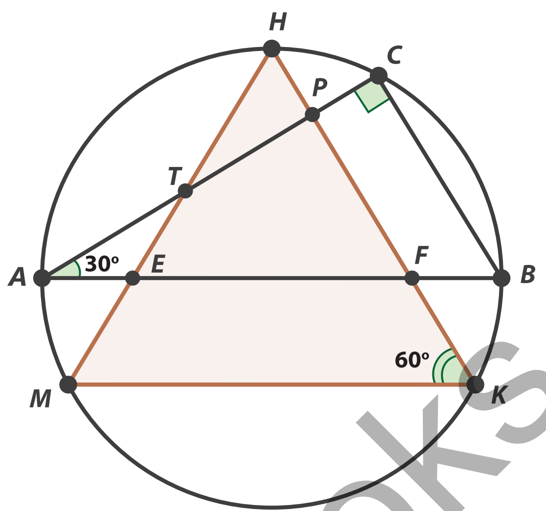
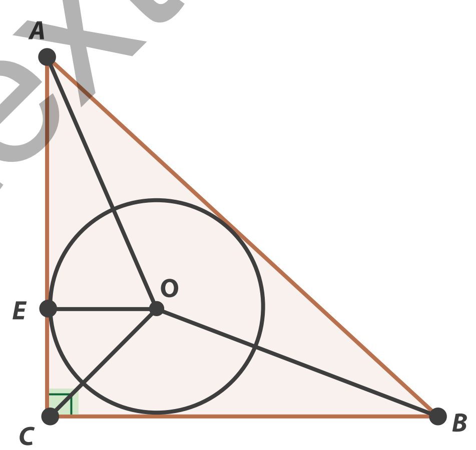
Explain why.

5. Given two circles $\omega_1(O, R=9)$, $\omega_2(P; R=15)$. How can these circles be positioned relative to one other, if the distance between their centres is equal to:

a) 0; b) 30; c) 24; d) 20;

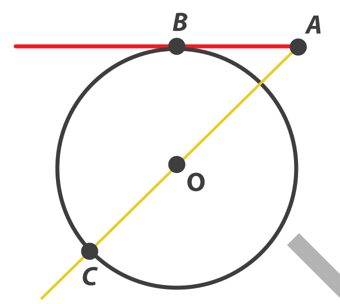
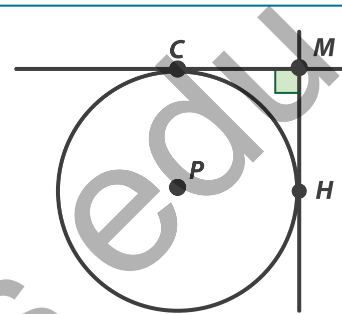
6. Aliya drew two circles $\omega_1(O, R)$, $\omega_2(P, R=12)$ and a segment OP = 24. She drew a perpendicular bisector *a* to the segment OP. What can you say about the relative position of line *a* and circle ω_2 ? What is the radius of ω_1 , if the circle does not have common points with the line *a*?

7. Work with the drawing and answer the questions. Explain your answers.

1. $\triangle ABC(\angle C = 90°)$, $\triangle MHK$ $(MH=HK=MK)$, $AB \parallel MK$, $\angle A = 30°$. 	a) Where is the centre of the circle located? b) How are the chords *CB* and *KH* located? c) How are the chords *AC* and *MH* located? d) What is type of the triangle *HPT*? e) What is type of the triangle *ATE*?
2. $\triangle ABC(\angle C = 90°)$, $AC = CB$. 	a) What is the angle of the *COB*? b) What is type of the triangle *AOB*? c) What do the angles of the triangle *EOC* equal to? d) How are the lines *EO* and *CB* positioned?

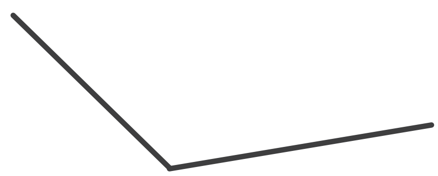
5.15 Problem solving

1. Solve the problems:

a) Given a circle with a tangent AB and a transversal AC drawn from a point A. Find the angles of triangles ABO and ABC if $AB = BO$.	
б) Two tangents MC and MH are drawn from the point M to the circle $\omega(P; R = 11)$ so that $MC \perp MH$. Find the lengths of the segments MC and HP.	

2. Given incircle of an isosceles triangle ABC. Find the perimeter of the triangle ABC if the tangency point divides the side into segments equal to 6 cm and 8 cm.

3. Use a compass and a ruler to construct:

a) an angle equal to the given one; b) a circle that is tangent to the sides of a given angle and with a centre at 5 cm distance from the vertex of the angle.	

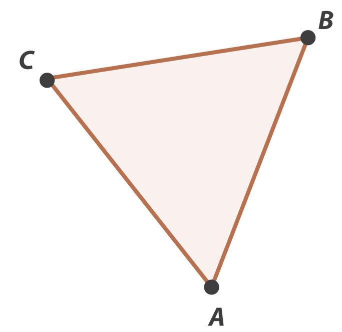
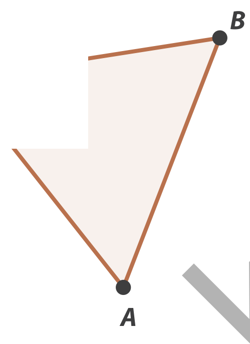
4. Use a compass and a ruler to construct:

a) a right triangle so that a given leg a is twice as large as leg b.	
b) a right triangle so that the leg a is a half of the given hypotenuse c.	

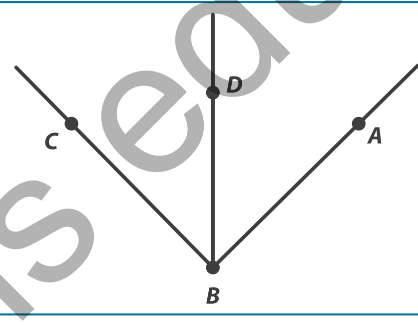
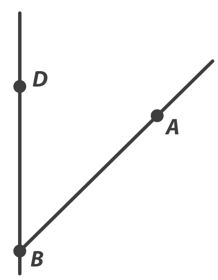
5. Given two parallel lines. Construct a third line that is equidistant from these lines.

6. Complete the drawing.

Arman drew an equilateral triangle, as shown in the drawing, and wiped out a part of the triangle. How can he restore the drawing?

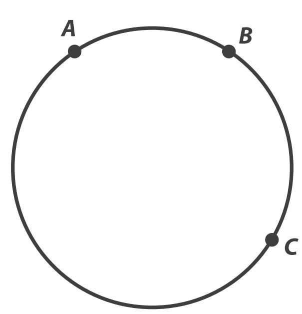
	

7. Restore the drawing.

Arman constructed an angle *ABC* and drew a bisector *BD*.	
Then he wiped out a half-line *BC*.	
Then he wiped out a half-line *BA*.	

Restore the angles

8. Arman built a circle, but accidentally wiped out its centre. Can you restore it?



5.16 What do I know?

Fill in the gaps to revise your knowledge about a disk, a circle, construction problems and locus of points.

Circle, disk, construction problems, LOP

Circle, disk and their elements

Circle is ....

Disk is ....

Radius of a circle is ....

Radius of a circle is ....

Chord is a segment ....

Arc of a circle is ....

Sector of a circle is ....

If a diameter is perpendicular to a chord, then ....

Relative position of a line and a circle

A circle and a line *do not intersect* if ....

A line and a circle have *one common point*, if ....

A line and a circle have *two common points* if ...

Transversal is ....

Tangent is ....

The relative position of two circles

Two circles *do not intersect* if ....

Two circles *are tangent* if ....

Two circles *are tangent externally* if ....

Two circles *are tangent internally* if ....

Two circles *intersect* one other if ....

Incircles and circumcircle

A circle is *inscribed* in a triangle if ....

A circle is *circumscribed* about a triangle if ....

The centre of the incircle of a triangle is ....

The centre of the circumcircle of a triangle is ... ..

LOP. Construction problems

Locus of points is ... .

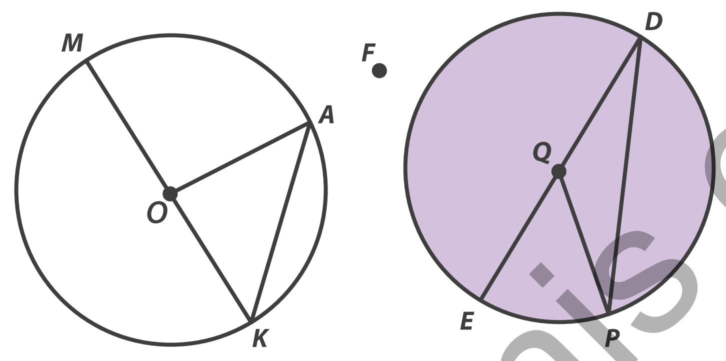
Construction problems include ... .

Questions that will help you to revise your learning.

Make sentences using the following words at least once:

- compass;
- circumcircle;
- incircle;
- sector;
- chord;
- segment;
- radius;
- circle;
- disk.

1. Work with the drawings and fill in the table



Drawing 1

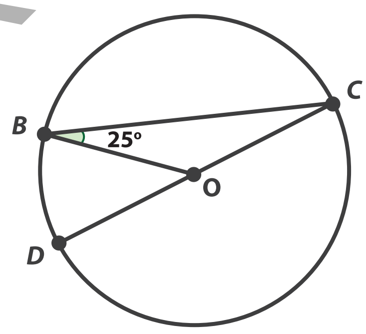
Drawing 2

	radius	diameter	chord	centre	sector	segment
Circle						
Disk						

2. Damir drew a circle with the centre *O* and drew diameters *MN* and *KP*. Is it true that *MK* = *NP*? Explain your answer.

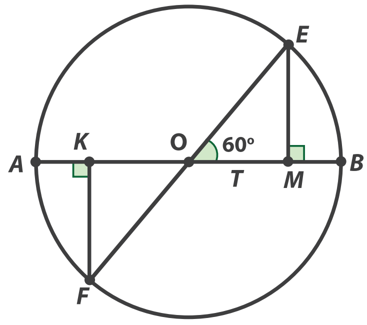
3. Solve the problems using ready made drawings:

a)



Find: $\angle BDO$.

b)



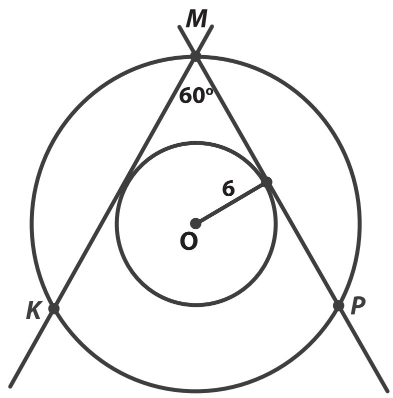
Given: *KM*=15.
Find: *FE*.

4. The lateral side of an isosceles triangle is divided by the tangency point of its incircle in a 3:2 ratio. Find a lateral side of the triangle if its perimeter is 81 cm. How many solutions does the problem have? Consider all possible cases.

5. Solve the problems using ready made drawings:

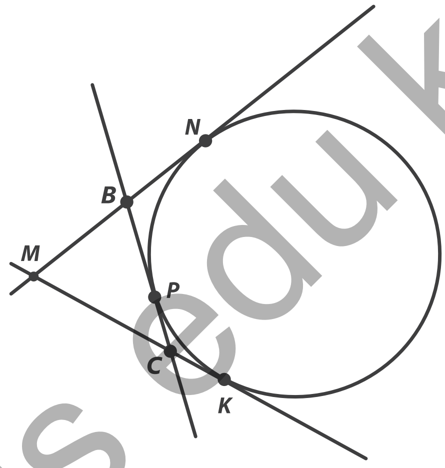
a) **Given:** $\omega_1(O;6)$, $\omega_2(O;r)$, $\angle KMP = 60°$.

Find: OK.



б) **Given:** $P(MBC) = 18$.

Find: MN.



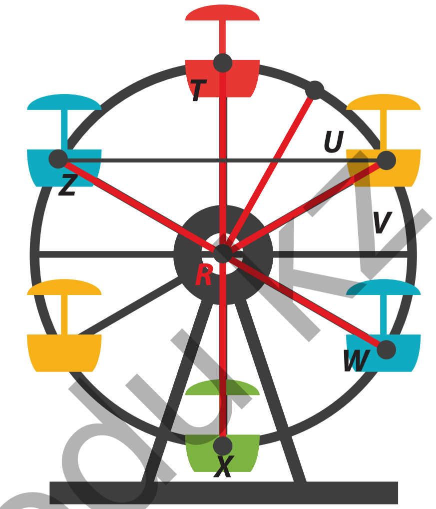
6. Construct using a compass and a ruler. Construct:

a) a right triangle with a given altitude drawn to a hypotenuse and a leg;
b) an equilateral triangle with a given median.

5.17 What have I learned? Self-assessment activities

1. Here is a mathematical model of a Ferris wheel.
List the following:

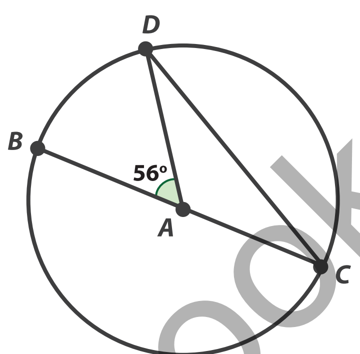
a) a centre of a circle;
b) a radius of a circle;
c) a diameter of circle;
d) a chord of a circle.



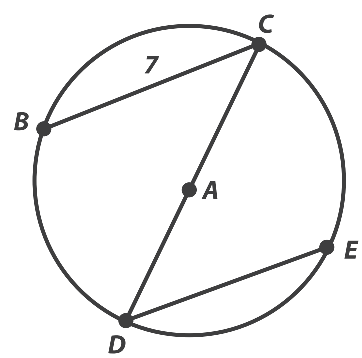
Find all kinds of triangles that can be inscribed in a given circle. Name right triangles.

2. Solve problems using ready-made drawings:

a) Given: $\angle BAD = 55°$.
Find: $\angle CDA$.



b) Given: $BC \parallel DE$, $BC = 7$.
Find: DE.



3. Use a compass and a ruler construct.
Construct an isosceles triangle with a given median drawn to the base and a side of the triangle.

4. Draw a circle of any radius. Mark points on the circle that are equidistant from a given line. How many solutions did you get?

6 Algebraic fractions

By the end of this unit, you will have learned:
- ✔ what fraction is called an algebraic fraction;
- ✔ what is a tolerance range of an algebraic fraction;
- ✔ how to simplify an algebraic fraction.

I will be able to:
- ✔ find a tolerance range of variables of an algebraic fraction;
- ✔ use the basic property of algebraic fractions when transforming and simplifying;
- ✔ add, subtract, multiply and divide algebraic fractions;
- ✔ simplify the expressions containing algebraic fractions.

Algebraic expressions

NUMERIC

Find a common denominator: $\frac{1}{1+x^8}, \frac{1}{1+x^4}, \frac{1}{1+x^2}, \frac{1}{1+x}, \frac{1}{1-x}$.

Simplify the expression: $2-\cfrac{x}{x-\cfrac{x}{x+\cfrac{x}{x-\cfrac{x}{2}}}}$.

LITERAL

Find $\frac{a}{b}$, if $\frac{a+3b}{b}=5$.

Find all the values of the variable x, if $\frac{8-x}{x^2+5}=0$.

6.1 Algebraic fractions

You know that expressions are numeric and literal (sometimes they are called variable expressions), and together they are called algebraic expressions.

In turn, algebraic expressions can be integer and fractional. Fractional algebraic expressions are called algebraic fractions.

1. Marat wrote algebraic expressions on the left and right sides of the board:

$$\frac{1}{2}x,\ \frac{c}{5},\ \frac{2+b}{3},\ y^2-2y+1. \qquad \frac{2}{x},\ \frac{5}{c},\ \frac{3}{2+b},\ \frac{1}{y^2-2y+1}.$$

What are the similarities and differences between these expressions? Which of these expressions are integers and fractional? Why do you think so? Explain your answer.

An integer algebraic expression is an algebraic expression that contains only addition, subtraction and multiplication operations.

A fractional algebraic expression is an algebraic expression that contains division into a literal expression.

2. Which of the following expressions are integer and fractional algebraic expressions? Why?

a) $(a-3)^3-\frac{2}{a}$; b) $\frac{2a^2}{7}+\frac{3}{4}$; c) $6by^5$; d) $\frac{2}{3z-4}$; e) $\frac{x-3}{x+3}$; f) $\frac{cz}{4}-d$;

g) $\frac{2}{3}a^2b^3$; h) $\frac{x^4-5xy}{7}$; i) $\frac{5x^3}{9y^5}$; j) $\frac{1}{6}x^3y$; k) $\frac{7}{a^2-b^2}$; l) $(x-y)^2-4xy$.

An algebraic expression that is a fraction with polynomials in the numerator and denominator is called an algebraic fraction.

3. Are these algebraic expressions algebraic fractions? Why? Explain your answer.

a) $\frac{3}{b}$; b) $\frac{x+y}{z}$; c) $\frac{x^2+y^2}{6}$; d) $\frac{x-y}{5}-a$; e) $\frac{a(x-y)}{b(x+y)}$; f) $\frac{a^3+b^2-4}{c^3+6}$; g) $\frac{\frac{m+n}{p}}{2m^3}$; h) $\frac{\frac{a}{b}+c}{c-\frac{a}{b}}$.

4. Write an algebraic fraction:

a) the numerator of which equals the sum of squares of variables x and y; the denominator of which equals the sum of these variables;
b) the numerator of which equals the difference of cubes of variables a and b; the denominator of which equals the sum of cubes of these variables;
c) the numerator of which equals the square of difference of variables m and n; the denominator of which equals the doubled product of these variables.

An algebraic fraction is an expression of the form $\frac{A}{B}$,

where A and B - polynomials, and $B \neq 0$

Algebraic expression

is an algebraic fraction	***is not** an algebraic fraction*
$\frac{a+b}{a-b}$	$\frac{a}{2}-c$
$\frac{m^2+n^2}{5m}$	$x+\frac{x-5}{7}$
	$x^2-\frac{y^3}{3}$

5. Find the value of the algebraic fraction with given variables:

a) $\frac{a-b}{9}$;	$a=2,5;\ b=-6,5;$	$a=\frac{7}{6};\ b=-\frac{5}{6};$
b) $\frac{3x+5y}{x-2}$;	$x=\frac{2}{3};\ y=-\frac{3}{5};$	$x=-11,2;\ y=5,6;$
c) $\frac{2m-3n}{n-5m}$;	$m=4,5;\ n=-\frac{2}{3};$	$m=\frac{1}{2};\ n=\frac{1}{3}.$

6. Find the value of the algebraic fraction and complete the table. Do you always can do this? Explain your answer.

		$a=-4$	$a=-3$	$a=0$	$a=3$	$a=4$	$a=9$
а)	$\frac{5}{a-3}$						
б)	$\frac{6a}{a-4}$						
в)	$\frac{a-3}{a^2-9}$						

7. Create a mathematical model to find:
a) the area of a square with a side equal to $3x+1$ meters;
b) the sides of a rectangle with the area of S dm^2 and the length of the second side equal to x dm;
c) the perimeter of a triangle with the sides equal to $2x-5$, $3x+1$ and $4x+2$.

6.2 Algebraic fractions. Problem solving

An algebraic fraction is meaningful when its denominator is different from zero. Values of variables at which the algebraic fraction has meaning are called permissible values of a variable.

1. Find permissible values of the variable *x* for the following algebraic fractions:

a) $\dfrac{3}{x-3}$; б) $\dfrac{x}{x-6}$;

b) $\dfrac{x-2}{3x-12}$; г) $\dfrac{x+5}{5-x}$;

c) $\dfrac{3}{x^2-4}$; е) $\dfrac{6}{(x-1)(x+5)}$;

d) $\dfrac{3x}{4x^2-25}$; з) $\dfrac{6-x}{4-4x+x^2}$

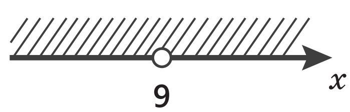
e) $\dfrac{8}{|x|-2}$; к) $\dfrac{7}{x-a}$.

Example:
permissible values for the following algebraic fraction
$$\frac{3}{(x-1)(x-5)}$$
are every value of *x*, except for *x*=1 and *x*=5.
This fraction does not exist, if *x*=1 and *x*=5.

2. Write an algebraic fraction that:
a) is meaningless when the value of a variable is 5;
b) is meaningless when the values of a variable are 0 and -3;
c) is meaningless at any value of a variable;
d) is meaningful at any value of a variable.

Set of permissible values of a variable that makes an algebraic fraction meaningful is a tolerance range of this fraction.

3. Complete the table:

	Expression	**Permissible values of a variable**	**Graphical representation of the set of permissible values of a variable:**
a)	$\dfrac{6}{x-9}$	$x \neq 9$	
b)	$\dfrac{x-9}{6}$		
c)	$\dfrac{x}{x+3}$		
d)	$\dfrac{x+3}{x}$		
e)	$\dfrac{x}{3x-6}$		

f)	$\dfrac{7}{(x-4)(x+5)}$		
g)	$\dfrac{4}{\lvert x\rvert-5}$		

4. Which of the following numbers -5, -4, 0, 4, 5, 16 are in the tolerance range of the algebraic fraction: $\dfrac{x-4}{(x^2-16)(x-5)}$ **?**

5. Find the tolerance range of an algebraic expression:

a) x^2-2x+3; b) $\dfrac{3}{x^2+4}$; c) $\dfrac{\frac{3}{x}-2}{x-3}$;

d $\dfrac{4x}{(x-4)(x+6)}$; e) $\dfrac{36}{\lvert x\rvert-6}$; f) $\dfrac{9}{\lvert x\rvert+9}$.

6. At which values of a variable the value of an algebraic fraction:

a) $\dfrac{x+3}{7}$ is 5; b) $\dfrac{p-7}{4}$ is -6;

c) $\dfrac{6}{y-3}$ is 4; d) $\dfrac{5}{3+z}$ is -9?

An algebraic fraction is equal to zero, if the numerator of the fraction is zero.
$\dfrac{A}{B}=0$, if $A=0$ and $B\neq 0$,
where A and B are polynomials.

7. At which values of a variable the value of an algebraic fraction is equal to zero?

a) $\dfrac{6x}{x-2}$; b) $\dfrac{3}{x-7}$; c) $\dfrac{b^2-25}{b-3}$; d) $\dfrac{x^2+4}{x+4}$; e) $\dfrac{b^2-25}{b-5}$;

f) $\dfrac{x-3}{x^2-9}$; g) $\dfrac{x+4}{x^2+4}$; h) $\dfrac{x-2}{\lvert x\rvert-2}$; i) $\dfrac{2x}{x^2-6x}$;

8. Solve the equation:

a) $\dfrac{x-5}{64}=0$; b) $\dfrac{x-8}{x}=0$; c) $\dfrac{4x-9}{x-9}=0$;

d) $\dfrac{x^2-25x}{x}=0$; e) $\dfrac{x^2+4}{x}=0$; f) $\dfrac{2x-4}{x^2-4}=0$.

9. At which values of a variable, the following statements are true:

a) $\dfrac{m}{n}=0$; b) $\dfrac{m}{n}=1$; c) $\dfrac{m}{n}=-1$; d) $\dfrac{m}{n}<0$; e) $\dfrac{m}{n}>0$?

6.3 Basic property of fraction

You have already studied ordinary fractions, and you know that they have a number of properties. Let us see if we can apply these properties to algebraic fractions.

1. "Find a pair." For each fraction written in the first line of the table, find an equal fraction from the second line. What property of fraction did you apply?

$\frac{3}{18}$	$\frac{6}{7}$	$\frac{4}{36}$	$\frac{2}{7}$	$\frac{125}{325}$
$\frac{1}{9}$	$\frac{18}{63}$	$\frac{5}{13}$	$\frac{72}{84}$	$\frac{1}{6}$

$\frac{b}{2b}$	$\frac{cd}{ce}$	$\frac{a}{m}$	$\frac{k}{2}$	$\frac{2cb^3}{4b^2}$
$\frac{cb}{2}$	$\frac{a^2}{am}$	$\frac{(m+n)k}{2(m+n)}$	$\frac{d}{e}$	$\frac{1}{2}$

Basic property of fractions that is true for algebraic fractions:

If the numerator and denominator of an algebraic fraction are multiplied or divided by the same polynomial other than zero, the fraction be equal to the given draction.

Basic property of algebraic fraction:

$$\frac{A}{B}=\frac{A\cdot C}{B\cdot C}=\frac{AC}{BC},$$

A, B, C are polynomials, $B\neq 0$, $C\neq 0$.

2. Fill the gaps to get correct equalities:

а) $\frac{3x}{7y}=\frac{\dots}{35y}=\frac{3x^2}{\dots}$;	б) $\frac{x}{4y^2}=\frac{ax}{\dots}=\frac{\dots}{4b^2y^2}$;	в) $\frac{a-b}{a+b}=\frac{\dots}{a^2-b^2}=\frac{a^2-b^2}{\dots}$.

The main property of fraction is the basis for bringing the fractions to a common denominator. Let us consider how to bring algebraic fractions to a common denominator.

3. Comment on how the given fractions were reduced to a common denominator:

$\frac{1^{\backslash 1}}{36}$ and $\frac{5^{\backslash 2}}{18}$;	$\frac{2^{\backslash a}}{ab^2}$ and $\frac{5^{\backslash b}}{a^2b}$;	$\frac{2^{\backslash a+b}}{a+b}$ and $\frac{5^{\backslash 1}}{(a+b)^2}$;
$\frac{1}{36}$ and $\frac{10}{36}$.	$\frac{2a}{a^2b^2}$ and $\frac{5b}{a^2b^2}$.	$\frac{2(a+b)}{(a+b)^2}$ and $\frac{5}{(a+b)^2}$.

4. Reduce the following fractions $\frac{7y}{8x^2}$, $\frac{5}{2xy}$, $\frac{3}{4x^2y^2}$, $\frac{8x}{y^3}$ **to the denominator** $16x^3y^4$.

Algorithm for bringing algebraic fractions to a common denominator:

1. Factor the denominators of fractions.
2. Find the common denominator of the fractions.
3. Find an additional factor for each fraction.
4. Multiply the numerator of each fraction by its additional factor.
5. Write each fraction with the found numerator and common denominator.

5. Use the algorithm and bring algebraic fractions to a common denominator:

a) $\frac{3x}{5}$ and $\frac{2y}{3}$; b) $\frac{12}{ab}$ and $\frac{-16}{3b^2}$; c) $\frac{6}{5x}$; $\frac{3x}{2a}$ and $\frac{-8}{3b}$; d) $\frac{x}{3y^2}$ and $\frac{x^2}{4y^3}$; e) $\frac{2}{-5a^2b}$; $\frac{4}{15ac}$ and $\frac{-3}{20ab^2}$; f) $\frac{8}{x-y}$ and $\frac{9}{y-x}$; g) $\frac{2a}{5a+4b}$; $\frac{3b}{4b-5a}$ and $\frac{4a}{25a^2-16b^2}$;

h) $\frac{5}{a^2-b^2}$ and $\frac{3}{a^2+2ab+b^2}$.

6. Represent the algebraic expressions in the form of fractions with the same denominators:

a) $\frac{1}{(x-y)(y-z)}$, $\frac{1}{(y-z)(x-z)}$ and $\frac{1}{(z-x)(y-x)}$; b) $\frac{3x}{4x^2-1}$ and $\frac{x+1}{2x^2+x}$;

c) $\frac{1}{x-y}$, $\frac{xy}{x^2-y^2}$ and $\frac{2x}{x^2-2xy+y^2}$; d) $\frac{2}{a^2-2ab+b^2}$ and $\frac{4}{a^2-b^2}$;

e) $\frac{2}{x-y}$, $\frac{3xy}{x^3-y^3}$ and $\frac{5y}{x^2+xy+y^2}$; f) $\frac{6}{x-y}$, $\frac{3}{x^2-y^2}$ and $\frac{6}{x^3-y^3}$,

g) $\frac{x-y}{x^2+xy}$, $\frac{x}{y^2+xy}$ and $\frac{y^2}{x^3-xy^2}$; h) $\frac{3x+y}{9x^2-24xy+16y^2}$, $\frac{2xy}{9x^2+24xy+16y^2}$ and $\frac{3y}{9x^2-16y^2}$.

7. Use the formulas for abridged multiplication and bring the fractions to a common denominator:

$$\frac{1}{1+x^8}, \frac{1}{1+x^4}, \frac{1}{1+x^2}, \frac{1}{1+x}, \frac{1}{1-x}.$$

8. Find the value of a variable or express *a*:

a) $\frac{5}{9}=\frac{-a}{27}$; b) $\frac{-7}{13}=-\frac{a}{52}$; c) $\frac{-y^2}{c}=\frac{a}{yc}$; d) $-\frac{x}{b}=\frac{x^2}{a}$; e) $\frac{-xy}{x^3z}=-\frac{y}{a}$.

9. *A* and *B* are polynomials. Are the following equalities correct? Why? Explain your answer.

a) $\frac{-A}{B}=-\frac{A}{B}$; b) $\frac{-A}{B}=-\frac{A}{-B}$; c) $\frac{A}{B}=-\frac{-A}{-B}$; d) $\frac{A}{B}=-\frac{-A}{B}$.

10. Arman says that the algebraic fractions are equal to the following equalities:

$\frac{A-B}{C-D}=\frac{B-A}{D-C}=-\frac{A-B}{D-C}=-\frac{B-A}{C-D}$. **Is Arman right? Explain your answer.**

6.4 Reduction of algebraic fractions

Use the basic property of algebraic fractions, so you can bring a fraction to a common denominator, and also reduce algebraic fractions. First of all, you need to factor the numerator and denominator (if possible) and then make possible reductions.

What factoring techniques do you know?

1. Reduce the algebraic fractions:

a) $\frac{38}{76}$; b) $\frac{16m}{64n}$; c) $\frac{6(m+1)}{54(m+1)}$; d) $\frac{5(a+b)}{-7(a+b)}$; e) $\frac{6x(y+z)}{11(y+z)}$; f) $\frac{4m(x+y)}{12m(x+y)(x-y)}$;

g) $\frac{3(a-b)}{b-a}$; h) $\frac{20(x-y)}{15(y-x)}$; i) $\frac{(n-m)^2}{m-n}$; g) $\frac{a-b}{(a-b)^2}$; k) $\frac{7x^2(y-z)}{35x^3(y-z)^2}$; l) $\frac{27a^3b^2(a-b)}{81a^2b^4(a-b)^2}$.

2. Reduce the fraction:

a) $\frac{125a^5b^4}{25a^3b^6}$; b) $\frac{125(-a)^5b^4}{25a^3b^6}$; c) $\frac{3ab}{27a}$; d) $\frac{4y^3}{12yz}$; e) $-\frac{7xy}{63x^3y^2}$; f) $\frac{81m^4n^3}{27m^3n^2}$.

3. Erlan wrote the examples of reducing fractions on the board. Comment on his solutions. Did he do everything right?

a) $\frac{\cancel{m}n-k}{\cancel{m}t}=\frac{n-k}{t}$; b) $\frac{x^3+4x}{7x^2-6x}=\frac{x+2}{7-6}=x+2$.

4. Take out the common factor and reduce the fraction:

a) $\frac{7x}{14x-21y}$; b) $\frac{8x-16a}{12ax+32a}$;

c) $\frac{a^3-5a^2b}{a^2x+a^3y}$; d) $\frac{x-12xy}{x^2y-12x^2y^2}$;

5. Simplify the expression:

a) $\frac{x^2-16}{4-x}$; b) $\frac{m^2-64}{(m-8)^2}$; c) $\frac{x^2-2x+1}{x-1}$;

d) $\frac{a^2+10ab+25b^2}{a^2-25b^2}$; e) $\frac{x^3-y^3}{cx-cy}$; f) $\frac{x^2+xy+y^2}{ax^3-ay^3}$.

Example:
Simplify the expression

$$\frac{a^2-2ab+b^2}{b^2-a^2}.$$

$$\frac{a^2-2ab+b^2}{b^2-a^2}=\frac{(a-b)^2}{(b-a)(b+a)}=$$

$$=\frac{(b-a)^2}{(b-a)(b+a)}=\frac{b-a}{b+a}=\frac{b-a}{a+b}.$$

6. Prove that the value of an algebraic fraction does not depend on variables *a* and *b*:

a) $\frac{ax-ay+bx-by}{a+b}$; b) $\frac{3a+3b}{3a+3b+ax+bx}$; c) $\frac{ax-ay-bx+by}{ax+ay-bx-by}$.

7. Reduce the fractions. Which factoring techniques did you use?

a) $\frac{m^2-n^2}{m^2+mn}$; b) $\frac{5x^2-20x}{x^2-8x+16}$;

c) $\frac{ax-ay-bx+by}{x-y}$; d) $\frac{p^2-q^2}{5p-2p^2+5q-2pq}$.

8. Reduce the fractions:

a) $\frac{\frac{1}{2}x+\frac{1}{3}y}{\frac{2}{3}y^2+2xy+\frac{3}{2}x^2}$; b) $\frac{\left(\frac{1}{48}m-\frac{1}{36}n\right)\left(\frac{1}{48}m+\frac{1}{36}n\right)}{\frac{1}{16}m^2-\frac{1}{9}n^2}$.

Example:

$$\frac{\frac{1}{3}a-\frac{1}{2}b}{\frac{1}{4}a^2-\frac{1}{6}ab+\frac{1}{3}b^2}=$$

$$=\frac{12\left(\frac{1}{3}a-\frac{1}{2}b\right)}{12\left(\frac{1}{4}a^2-\frac{1}{6}ab+\frac{1}{3}b^2\right)}=$$

$$=\frac{4a-6b}{3a^2-2ab+4b^2}.$$

9. Arman says that he can write five fractions in five minutes:

a) with a denominator a-3, and they will be reducible;
b) that will result in the following fraction after reduction $\frac{2}{a-2}$;

Can you do the same?

10. Reduce the fraction and find its numeric value:

a) $\frac{a^2-14a+49}{ab-7b}$, if $a=-2$, $b=3$; b) $\frac{a^2-2a-15}{a^2+6a+9}$, if $a=5\frac{1}{2}$;

c) $\frac{x^3-x^2y+xy^2}{x^3+y^3}$, if $x=\frac{2}{5}$, $y=0,2$.

11. Reduce the fractions:

a) $\frac{6x^3}{|x|}$; **b)** $\frac{3|x|}{x^3}$; **c)** $\frac{x-5}{|x|-5}$.

12. At which values of *p*, the fraction will be reducible $\frac{y^2-81}{y-p}$**?**

6.5 Operations with algebraic fractions

$$\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C};$$

$$\frac{A}{C}-\frac{B}{C}=\frac{A-B}{C},$$

where A, B, C are polynomials, $C \neq 0$.

As you know, when adding ordinary fractions with the same denominators, we add their numerators, but the denominators remain unchanged.

Adding and subtracting algebraic fractions with the same denominators is carried out according to the same rules as adding and subtracting ordinary fractions.

1. Marat solved some examples. Are they solved correctly? Explain your answer.

a) $\frac{x}{z}+\frac{y}{z}=\frac{x+y}{z}$; **b)** $\frac{x-2}{3a}-\frac{y+5}{3a}=\frac{x-y+3}{3a}$; **c)** $\frac{m+n}{z+m}-\frac{m-n}{z+m}+\frac{m^2-n^2}{z+m}=\frac{m^2+2n-n^2}{z+m}$.

2. Carry out the operations:

a) $\frac{5a-3b}{c}-\frac{a-b}{c}$; b) $\frac{b+6}{a-5}+\frac{b-1}{a-5}$; c) $\frac{3x-4y}{12xy}+\frac{5x-4y}{12xy}$; d) $\frac{m}{x-3}+\frac{n}{3-x}$;

e) $\frac{x^2+16}{3x-12}-\frac{8x}{3x-12}$; f) $\frac{a-bx}{b}-\frac{a-2bx}{b}$; g) $\frac{x+y}{x^2-z^2}-\frac{x-y}{z^2-x^2}$; h) $\frac{10x-y}{a^3}-\frac{3x-y}{a^3}$.

3. Represent the following expressions as the sum and difference of algebraic fractions with the same denominator:

a) $\frac{4p-2q}{3}$; b) $\frac{2m+3n}{5k}$; c) $\frac{7x^2+5xy-2y}{x-y}$; d) $\frac{a^2-4a+8}{a}$.

4. Find the values of the expression with given values of a variable:

a) $\frac{x}{y-2}-\frac{7}{y-2}$, if $x=9$ and $y=2,5$; b) $\frac{x+4}{x^2-64}+\frac{4}{x^2-64}$, if $x=8,1$;

c) $\frac{x^2+16}{x-4}-\frac{8x}{x-4}$, if $x=4,5$; d) $\frac{x^2-3}{x-3}-\frac{6}{x-3}$, if $x=-3,5$;

e) $\frac{x^2+xy}{x^3-y^3}+\frac{y^2}{x^3-y^3}$, if $x=-0,5$ and $y=-0,25$.

5. Prove that the value of the expression $\frac{x^2}{x^2+3}-\frac{x^2-3}{x^2+3}$ is positive with any given values of a variable *x*.

6. Represent the fraction in the form of sum or difference of an integer expression and algebraic fraction:

a) $\dfrac{z+k^2}{k^2}$; b) $\dfrac{m^2+2mn-6}{m}$; c) $\dfrac{2a^2+8a-7}{a}$.

Examples:

$$\frac{a-7}{a}=\frac{a}{a}-\frac{7}{a}=1-\frac{7}{a}$$

$$\frac{a^2+3a-1}{a^2}=\frac{a^2}{a^2}+\frac{3a}{a^2}-\frac{1}{a^2}=1+\frac{3}{a}-\frac{1}{a^2}$$

7. Given $\dfrac{a+b}{a}=5$. **Find the value of the algebraic expression:**

а) $\dfrac{b}{a}$; б) $\dfrac{a}{b}$; в) $\dfrac{a-b}{b}$; г) $\dfrac{a-b}{a}$.

8. Find the natural values n, at which the fraction takes natural values:

а) $\dfrac{n+16}{n}$; б) $\dfrac{7n-5}{n}$; в) $\dfrac{n^2+4n-6}{n}$.

9. Translate this problem into mathematical language and create a mathematical model:

a) A cyclist has left point *A*. At the same time, the motorcyclist followed the cyclist from point *B*, which is located at a distance of m km from *A*. The cyclist was driving at a speed of *n* km/h and the motorcyclist at a speed of *p* km/h. What is the distance from point *A*, at which the motocyclist will catch up the cyclist?

b) The side of a square is *m* cm smaller than the side of a rectangle, and *n* cm larger than the other. Find the side of the square, if it is known that the area of the square is *p* cm less than the area of the rectangle.

6.6 Operations with algebraic fractions

Let us consider how to add and subtract algebraic fractions with different denominators.

1. Comment on the addition and subtraction of the algebraic fraction given below. What cases of addition and subtraction do you think should be considered?

$$\frac{n+m}{m}+\frac{m}{4n}=\frac{4n^2+4nm+m^2}{4mn}=\frac{(2n+m)^2}{4mn}$$

$$\frac{a^2}{a^2-4}+\frac{a^{/a-2}}{2+a}=\frac{a^2+a^2+2a}{a^2-4}=\frac{2a^2+2a}{a^2-4}$$

$$\frac{5}{a^2-ab}-\frac{5}{b^2-ab}=\frac{5^{/b}}{a(a-b)}-\frac{5^{/a}}{b(a-b)}=\frac{5b-5a}{ab(a-b)}=\frac{-5(a-b)}{ab(a-b)}=-\frac{5}{ab}$$

Make an algorithm for adding and subtracting algebraic fractions with different denominators.

a) To add two algebraic fractions with different denominators, you have to… .
b) To subtract two algebraic fractions with different denominators, you have to … .

$$\frac{A}{B}+\frac{C}{D}=\frac{AD+BC}{BD}\quad \frac{A}{B}-\frac{C}{D}=\frac{AD-BC}{BD},$$

where A, B, C are polynomials,, $B\neq 0$, $D\neq 0$.

2. Find the sum or difference of the algebraic expressions:

a) $\frac{7}{8}+\frac{3}{4}$; b) $\frac{1}{32}-\frac{5}{16}$; c) $\frac{m}{n}+\frac{n}{m}$; d) $\frac{k}{p}-\frac{1}{2}$;

e) $\frac{a}{xy}+\frac{a}{xz}$; f) $\frac{3a}{mx}-\frac{4b}{nx}$; g) $\frac{5a}{18b}+\frac{6a}{81b}$; h) $\frac{7}{8x^3y^4}+\frac{5}{6x^4y^3}$;

i) $\frac{9x}{14a^3}-\frac{8y}{21a^5}$; j) $\frac{a}{a-b}-\frac{b}{a+b}$; k) $\frac{a}{a-b}+\frac{b}{a+b}$; l) $\frac{x^2}{x-0,5}-2$.

3. Find the sum of the algebraic fractions and complete the table:

+	$\frac{2}{x+y}$	$\frac{6x}{x-y}$	$\frac{3xy}{x^2-y^2}$
$\frac{5xy}{x+y}$			
$\frac{x}{x^2-y^2}$			
$\frac{2y}{x^3-y^3}$			

REMEMBER!

If the denominators of algebraic fractions contain polynomials, use factorization of polynomials used to find the lowest common denominator.

4. Simplify the expression:

a) $\frac{a^2-b^2}{3}-\frac{(a-b)^2}{12}-\frac{(a+b)^2}{4}$; b) $\frac{5z}{6x^2y}-\frac{7x}{8zy^2}+\frac{11}{12xz^2}$; c) $\frac{3x}{a^2-4}+\frac{4x}{a-2}-\frac{5x}{a+2}$

5. Fill the gaps:

a) $\frac{8}{\dots}-\frac{5}{x(x+1)}=\frac{\dots}{x(x^2-1)}$; b) $\frac{y-6}{2y-y^2}+\frac{6}{\dots}=\frac{\dots}{y(4-y^2)}$; c) $\frac{\dots+a+\dots}{2ab}=\frac{1}{\dots}+\frac{1}{2a}+\frac{1}{ab}$.

6. "Find a pair." Simplify the expressions and highlight the example and answer with the same colour.

$\frac{a^2}{ax-x^2}+\frac{x}{x-a}$	$\frac{x}{a+x}$	$\frac{2x^2}{x^2-a^2}-\frac{x}{x-a}$	$\frac{x^2+y^2}{x^3+y^3}-\frac{1}{2(x+y)}$
$\frac{1}{x^2+xy+y^2}+\frac{y}{x^3-y^3}$	$\frac{2a-y}{x-a}-\frac{a-y}{x-a}$	$\frac{a}{x-a}$	$\frac{x}{x^2+xy+y^2}$
$\frac{a+x}{x}$	$\frac{x}{a-x}$	$\frac{6x^2}{a-x^2}-\frac{2x}{a-x}+\frac{3x}{a+x}$	$\frac{x^2+xy+y^2}{x^3+y^3}$

7. Simplify and find a value of the expression:

a) $\frac{2}{a+2}+\frac{18}{2-a}+\frac{8}{a^2-4}$, if $a=-3$;

b) $\frac{x-1}{3x^2+6x+3}+\frac{1}{2x+2}$, if $x=-1$.

8. Translate the problem into mathematical language and make a mathematical model:

Yerzhan and his parents were returning home from the village where his grandmother lives. First, they drove a kilometer by bus at a speed of v km/h, then b kilometers by train at a speed of m times greater. How many hours were Yerzhan and his parents on the road?

9. The rocket.

Find the sum of two expressions, write the result in the third cell. Then find the sum of the last two expressions and write the result to the next cell. What expression will be in the fifth cell?

$\frac{1}{a+1}$	$\frac{1}{a-1}$			

6.7 Operations with algebraic fractions. Problem solving

As you have seen, the rules of addition and subtraction of numeric fractions are true for algebraic fractions. Do you think the rules of multiplication and division of numeric fractions are true for algebraic fractions? Let us talk about it in more detail.

1. Finish the sentences:

a) To multiply one numeric fraction by another, you have to ... ;
b) Inverse fractions are ... ;
c) To find a quotient of two numeric fractions, you have to ... ;
d) To raise an algebraic fraction to nth power, you have to ... .

2. Aliya multiplied and divided algebraic fractions. Comment on her solutions. How else can you perform this task?

a) $\frac{1}{2xy}\cdot\frac{6x^3y^2}{5z}\cdot\frac{15z^3}{2x^2}=\frac{1\cdot 6x^3y^2\cdot 15z^3}{2xy\cdot 5z\cdot 2x^2}=\frac{90x^3y^2z^3}{20x^3yz}=\frac{9yz^2}{2}$;

b) $\frac{4x^2-4x+1}{4x-2}:\frac{6x-3}{2x+1}=\frac{4x^2-4x+1}{4x-2}\cdot\frac{2x+1}{6x-3}=\frac{(2x-1)^2(2x+1)}{2(2x-1)\cdot 3(2x-1)}=\frac{2x+1}{6}$.

Derive the rules for multiplication and division of algebraic fractions.

Rules for multiplication, division and raising algebraic fractions to nth power.

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{A\cdot C}{B\cdot D}=\frac{AC}{BD},\ B,\ D\neq 0;\quad \frac{A}{B}:\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{AD}{BC},\ B,\ C,\ D\neq 0;$$

$\frac{A}{B}:\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{AD}{BC},\ B,\ C,\ D\neq 0;$ $\left(\frac{A}{B}\right)^n=\frac{A^n}{B^n}$, $B\neq 0$, where A, B, C, D are polynomials.

3. Perform the following operations:

a) $\frac{8}{15}\cdot\frac{27}{20}$; b) $\frac{15}{7}:\frac{25}{14}$; c) $\frac{x^2}{8a}\cdot\frac{24ab}{x^3}$; d) $\frac{45x^2}{a}\cdot\frac{x^3}{5a^4}$;

e) $\frac{4pq^2}{21a^2b}\cdot\frac{7ab^2}{16p^2q}$; f) $\frac{12x}{25y^2}:\frac{16x^4}{15y^3}$; g) $\frac{15xy}{64z}:25y^2$; h) $12m^2n:\frac{3m^3}{4n^2}$;

i) $\frac{x^2-y^2}{125xy^2}:\frac{x-y}{5xy}$; g) $\frac{a^3-ab^2}{27a^2}\cdot\frac{9a}{a-b}$; k) $\frac{17(x-y)}{4(x^2+y^2)}:\frac{(x-y)^3}{x^2+y^2}$; l) $\frac{x^2-y^2}{125x^2y^2}:\frac{x-y}{5xy}$.

4. Simplify the expression. Which rules did you use when working with algebraic fractions?

a) $\frac{x-3}{x^2+8x+16}\cdot\frac{(x+4)^2}{x^2-9}$;

b) $\frac{y^2-12y+36}{y+5}:\frac{(y-6)^2}{y^2-25}$;

c) $\frac{ax-ay}{bx+by}:\frac{bx-by}{cx+cy}$;

d) $\frac{x+y}{x^2+xy+y^2}\cdot\frac{x^3-y^3}{x^2-y^2}$;

e) $\frac{a^2+2a+1}{9a+1}:\frac{a+1}{81a^2-1}$;

f) $\frac{a^3+3a^2b+3ab^2+b^3}{a-b}\cdot\frac{a^2-b^2}{(a+b)^3}$.

5. Aliya wrote two equalities on the board and thinks they are correct. Is Aliya right? Why? Explain your answer.

$\frac{a^4-b^4}{c^2-d^2}\cdot\frac{c^3-c^2d}{a^2+b^2}=\frac{c(a^2-b^2)}{c+d}$, $c\neq d, c\neq -d$;	$\frac{x^2-xy}{x^2+y^2}:\frac{x^2y-xy^2}{x^4y-y^5}=\frac{1}{x^2-y^2}$, $x\neq y, x\neq -y$.

6. Perform the given operations with algebraic fractions *A*, *B* and *C*, and fill the table:

	A	B	C	$A+B$	$A-C$	$A+B-C$	$(A-B)\cdot C$	$(A+C):B$	$AB:C$
a)	$\frac{xy}{z}$	$\frac{x^2y}{z^2}$	$\frac{x^3y^3}{z^3}$						
b)	$\frac{x}{x-y}$	$\frac{y}{x+y}$	$\frac{xy}{x^2-y^2}$						
c)	$\frac{a-b}{3a}$	$\frac{2ab}{a+b}$	$\frac{ab}{a^2-b^2}$						

7. Express a variable of the formula, and complete the table:

a)	The area of a rectangle is $S=ab$, where a and b are the sides of the rectangle.	$a=\frac{S}{b}$	$b=\frac{S}{a}$
b)	The distance S travelled by a body at uniform motion is defined by the formula $S=vt$, where v is speed of the body and t is time of movement.	$v=$	$t=$
c)	Perimeter P of a rectangle is calculated by the formula $P=2(a+b)$, where a and b are the sides of the rectangle.	$a=$	$b=$
d)	Mass of a body P is calculated by the formula $P=mg$, where m is mass of the body, g is free fall acceleration.	$m=$	$g=$

8. Aliya wrote four polynomials on the board: a^2+ay, y^2, $a+y$, y. Use each of them only once, and write two fractions so that their product is equal to:

a) $\frac{a}{y}$; b) $\frac{y}{a}$; c) ay.

6.8 Simplifying expressions with algebraic fractions

You have already performed tasks to simplify the sum or difference, product or quotient of algebraic fractions. Let us consider how to simplify more complex expressions containing the operations with algebraic fractions.

1. Tanya and Marat simplified the expression in different ways. Tanya used the method of "*operations*", and Marat - method of "*chains*". They wrote their solutions on the board. Comment on the given solutions. Which method do you like better? Why? Explain your answer.

Method of "operations"

$$\left(\frac{1}{x+2}-\frac{4}{4x-x^3}\right):\left(\frac{x-2}{x^2+2x}-\frac{x}{2x+4}\right)=\frac{2}{2-x}$$

1) $\frac{1}{x+2}-\frac{4}{4x-x^3}=\frac{1}{x+2}-\frac{4}{x(2-x)(2+x)}=\frac{2x-x^2-4}{x(2+x)(2-x)}=\frac{-(x^2-2x+4)}{x(2+x)(2-x)}$;

2) $\frac{x-2}{x^2+2x}-\frac{x}{2x+4}=\frac{2x-x^2-4}{2x(2+x)}=\frac{-(x^2-2x+4)}{2x(2+x)}$;

3) $\frac{-(x^2-2x+4)}{x(2+x)(2-x)}:\frac{-(x^2-2x+4)}{2x(2+x)}=\frac{-(x^2-2x+4)}{x(2+x)(2-x)}\cdot\frac{2x(2+x)}{-(x^2-2x+4)}=$

$=\frac{-(x^2-2x+4)\cdot 2x(2+x)}{-x(2+x)(2-x)(x^2-2x+4)}=\frac{2}{2-x}$.

Method of "chains"

$$\left(\frac{1}{x+2}-\frac{4}{4x-x^3}\right):\left(\frac{x-2}{x^2+2x}-\frac{x}{2x+4}\right)=\left(\frac{1}{x+2}-\frac{4}{x(2-x)(2+x)}\right):\left(\frac{x-2}{x(x+2)}-\frac{x}{2(x+2)}\right)=$$

$$=\frac{-(x^2-2x+4)}{x(2+x)(2-x)}:\frac{-(x^2-2x+4)}{2x(2+x)}=\frac{-(x^2-2x+4)\cdot 2x(2+x)}{-x(2+x)(2-x)(x^2-2x+4)}=\frac{2}{2-x}.$$

2. Specify the order of operations and simplify the expression:

a) $a^2-\frac{a^3-4ab^2}{a^3-2a^2b+ab^2}\cdot\frac{a^2-2ab+b^2}{a-2b}$; b) $\left(\frac{a}{a+b}+\frac{a^2}{b^2-a^2}\right):\left(\frac{a^2}{a+b}-\frac{a^3}{a^2+b^2+2ab}\right)$;

c) $\frac{27-a^3}{3+a}:\left(3+\frac{a^2}{3+a}\right)-\frac{a^2}{a+3}\cdot\frac{9-a^2}{a^2-3a}$; d) $\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right):\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right)$;

e) $\left(\frac{2a^2+a}{a^3-1}-\frac{a+1}{a^2+a+1}\right)\cdot\left(1+\frac{a+1}{a}-\frac{a+5}{a+1}\right)$;

f) $\left(\frac{a^2-1}{a^2+a-6}:\frac{a^2-4a+3}{a^2-4}\right)\cdot\frac{a-3}{a^2+3a+2}$;

g) $\frac{a-2}{4a^2+16a+16}:\left(\frac{a}{2a-4}-\frac{a^2+4}{2a^2-8}-\frac{2}{a^2+2a}\right)$;

h) $\frac{a^2+2ab+b^2}{2ab}:\left(\frac{1}{a}+\frac{1}{b}\right)^2$;

i) $\left(\frac{2a}{a+2}+\frac{2a}{6-3a}+\frac{8a}{a^2-4}\right)\cdot\frac{3a-6}{2a^2}$.

REMEMBER!

Follow the order of operations to avoid errors.

3. Find the value of the algebraic expression:

a) $\left(\frac{x+y}{x-y}-\frac{x-y}{xy}\cdot\frac{xy}{x+y}\right):xy$, if $x=5\frac{3}{4}$, $y=3{,}75$;

b) $\left(\frac{2m^2+m}{m^3-1}-\frac{m+1}{m^2+m+1}\right)\cdot\left(1+\frac{m+1}{m}-\frac{m+5}{m+1}\right)$, if $m=2{,}25$;

c) $\frac{5}{4}-\frac{5a^2-10a+5}{4a^2+4a+4}:\frac{a-1}{a^3+a^2+a}$, if $a=\frac{1}{5}$.

4. Find the variable *x* from the proportion:

a) $\frac{a}{x}=\frac{5b}{6}$;

b) $\frac{x}{(a-b)^2}=\frac{a}{a-b}$;

c) $\frac{a+3}{a-3}=\frac{a^2-9}{ax}$;

d) $\frac{a^2-b^2}{x}=\frac{a+b}{a}$;

e) $\frac{x}{a^2-ab}=\frac{a-b}{a^2-b^2}$;

f) $\frac{a^2-2ab+b^2}{x}=\frac{a^2-b^2}{b}$.

5. Arman found a new method of quick calculations "in mind" based on the transformation of the expression $\frac{1}{x}-\frac{1}{x+1}$. What do you think is the method? Calculate the value of the expression using this method:

a) $\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$;

b) $\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}+\frac{1}{110}+\frac{1}{132}$.

Suggest your own examples for this method.

6. Translate the problem into mathematical language, and make a methematical model:

Aizhan took participation in a regatta and sailed S km for a period of time down the stream. How far will Aizhan sail upstream for the same period of time, if the speed of the stream is v_1 km/h, and the speed of the yacht is v km/h?

7. Fill the gaps with mathematical symbols and add brackets, so the equality is correct:

$$\frac{x}{y}\square\frac{x^2}{y^2}\square\frac{x^3}{y^2}=\frac{x+y}{x^2}.$$

6.9 Simplifying expressions with algebraic fractions. Problem solving

1. Place the cards in order.
Look at the result of the operations in each card. Find the result of the operations given on the first card, it will be used on the second card.

- **Start** $\frac{1}{5}$ multiplied by $\frac{x}{2}$
- $\frac{x}{10}$ divided by $\frac{x(m-n)}{10(m+n)}$
- **Finish** $\frac{(x^2+y^2)}{4x^2y^2}$ find the value of the expression, if $x = 2$, $y = 1$.
- 1 divided by the product of $\frac{x^3-y^3}{x^2+xy+y^2}$ and $(x + y)$
- x^2-y^2 added to $\frac{2y^4+2x^2y^2}{x^2+y^2}$
- $\frac{x(m-n)}{10(m+n)}$ multiplied by an inverse fraction
- x^2+y^2 divided by $2xy$ and raise to the second power

2. Simplify the expression:

a) $\left(\frac{m+n}{n-m}-\frac{n-m}{m+n}+\frac{6m^2}{m^2-n^2}\right):\left(\frac{m^2}{n^3-mn^2}+\frac{m-n}{n^2}-\frac{1}{n}\right)$;

b) $(x^2-y^2)\left(\frac{x-y}{x^2+y^2}+\frac{2x^2y+2xy^2}{5x^3+x^2y+5xy^2+y^3}\cdot\frac{5x+y}{x^2-y^2}\right)$.

3. Amina and Yerzhan have to simplify the algebraic fraction. They presented their solutions on the boards below.

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}=\frac{\frac{y-x}{xy}}{\frac{y+x}{xy}}=\frac{y-x}{xy}\cdot\frac{xy}{y+x}=\frac{y-x}{y+x}$$

Amina

$$\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}=\frac{xy\cdot\left(\frac{1}{x}-\frac{1}{y}\right)}{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}=\frac{y-x}{y+x}$$

Yerzhan

What did they use to simplify the expression? Who of them performed the task right? Which way would you choose to solve the problem? Why?

4. Simplify the expression:

a) $\dfrac{3-\dfrac{b}{y}}{3+\dfrac{b}{y}}$; b) $\dfrac{\dfrac{1}{a^2}-\dfrac{1}{2a^2}}{\dfrac{1}{a}-\dfrac{1}{2a}}$; c) $\dfrac{\dfrac{x^2+1}{x^2}+\dfrac{1}{x}}{x-\dfrac{1}{x^2}}$;

d) $\dfrac{\dfrac{2a-5}{b}-1}{\dfrac{2a}{b}+1}$; e) $\dfrac{\dfrac{x-y}{z}+3}{\dfrac{x+y}{z}-1}$; f) $\dfrac{y-\dfrac{ax}{x-a}}{x-\dfrac{ay}{y-a}}$;

g) $\dfrac{3+\dfrac{1}{a-1}-\dfrac{1}{a+1}}{a+\dfrac{a}{a^2-1}}$; h) $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{1}{1-x}-\dfrac{1}{1+x}}$; i) $\dfrac{x-\dfrac{k^2}{x}}{k-\dfrac{x^2}{k}}$.

5. Simplify the expression:

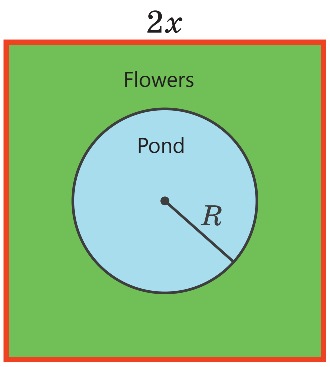
Example:

$$1-\frac{1}{1-\dfrac{x}{x+1}}=1-\frac{x+1}{(x+1)\left(1-\dfrac{x}{x+1}\right)}=1-\frac{x+1}{\left(x+1-\dfrac{x(x+1)}{x+1}\right)}=1-\frac{x+1}{x+1-x}=1-\frac{x+1}{1}=1-x-1=-x$$

a) $1+\dfrac{a}{1-\dfrac{a}{a+2}}$; б) $1-\dfrac{x}{1-\dfrac{x}{x+1}}$; в) $1+\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{a}}}$.

6. Make a mathematical model of the problem:

Gulnara decided to allocate a square plot of land $(2x \text{ x } 2x)$ m² in the garden and make a pond in the form of circle with a radius of R m. She decided to sow the lawn on an unoccupied part of the garden. How many seeds will she need and how much money will she spend to buy them?



Note!
To sow 1 m² of lawn, you need 40 g of seeds at average.
The cost of a 10 kg seed sack is 33 700 tg.

6.10 Simplification of expressions with algebraic fractions. Problem solving

1. Find the values of the expression with given values of variables. Will the equalities be always correct? Explain your answer.

$$\frac{1}{ab}=\frac{1}{a+b}$$

a	2	1	-3
b	3	4	2
$\frac{1}{ab}$			
$\frac{1}{a+b}$			

$$a\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{a}{b}+\frac{a}{c}$$

a	1	-2	
b	2	-3	
c	3	-4	
$a\left(\frac{1}{b}+\frac{1}{c}\right)$			
$\frac{a}{b}+\frac{a}{c}$			

$$\frac{7(a+1)^2}{a+1}=7(a+1)$$

a	1	-1	0
$\frac{7(a+1)^2}{a+1}$			
$7(a+1)$			

Identities are:

- the laws of arithmetic operations;
- rules of operations with powers and polynomials;
- formulas of abridged multiplication.

The equality that is correct at any permissible values of variables is called an identity.
The expressions written in the right and left parts of identity are identically equal.
The transition from one identity to another that is identically equal is called identity transformation.

2. Which equalities of the previous task are identities? At what values of variables the equalities can be identities? Explain your answer.

3. Which of these equations are identities? Why? Explain your answer.

a) $mn+mk=m(n+k)$; b) $m(nk)=mn+mk$; c) $x^2+x^3=x^5$; d) $x^2x^3=x^5$;

e) $(3y)^3=3y^3$; f) $(3y)^3=27y^3$; g) $(p+2)^2=p^2+4p+4$; h) $(p-2)^2=p^2-4$;

i) $\frac{a^4+a^2}{a^2+1}=a^2$; j) $\frac{a^4+a^2}{a^2+1}=2a^2$; k) $|2xy|=2xy$; l) $\frac{2|x|}{y}=\frac{2x}{|y|}$.

Other identities can be derived from known identities.
To prove whether an equality is an identity,
use the following techniques:

- perform identity transformations of the expressions on both sides of equality;
- prove that the difference between the right and left parts of equality is equal to zero.

Prove the identity: $\dfrac{4a^2+2ab+b^2}{8a^3-b^3}=\dfrac{1}{2a-b}$.	
$\dfrac{4a^2+2ab+b^2}{8a^3-b^3}=\dfrac{1}{2a-b}$, $\dfrac{4a^2+2ab+b^2}{(2a-b)(4a^2+2ab+b^2)}=\dfrac{1}{2a-b}$, $\dfrac{1}{2a-b}=\dfrac{1}{2a-b}$.	$\dfrac{4a^2+2ab+b^2}{8a^3-b^3}-\dfrac{1^{/4a^2+2ab+b^2}}{2a-b}=$ $=\dfrac{4a^2+2ab+b^2-4a^2-2ab-b^2}{(2a-b)(4a^2+2ab+b^2)}=0.$

4. Prove the identity:

a) $\dfrac{1}{x(x-1)}+\dfrac{1}{x(x+1)}+\dfrac{1}{(x-1)(x+1)}=\dfrac{3x}{x^2-1}$;	b) $\left(\dfrac{x+y}{x}-\dfrac{2y}{x+y}\right):\dfrac{x^2+y^2}{x^2-y^2}=\dfrac{x-y}{x}$;
c) $\dfrac{a}{a+2}+\left(\dfrac{1}{4-a^2}-\dfrac{1}{4-4a+a^2}\right):\dfrac{2}{(a-2)^2}=0$;	d) $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}+2}{\dfrac{x}{y}-\dfrac{y}{x}}=\dfrac{x+y}{x-y}$;
e) $\dfrac{\dfrac{1}{x-1}-\dfrac{4-x}{x^2-x}}{\dfrac{2}{x-1}-\dfrac{x+2}{x^2-x}}=2$;	f) $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{xy+xz+yz}=\dfrac{1}{xyz}$;
g) $\left(\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y}}-\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}\right)\cdot\left(\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{4}\right)=1$;	h) $\dfrac{\dfrac{1}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}:\left(\dfrac{\dfrac{1}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}-\dfrac{\dfrac{1}{b^2}+\dfrac{1}{a^2}}{\dfrac{1}{a^2}}\right)=\dfrac{1}{\left(\dfrac{a}{b}\right)^4}$;
i) $\dfrac{(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2}{(a-b)(b-c)(c-a)}=-1$.	

5. Write the equality of two expressions with the variable a. The left part of this equality should be defined for all values a different from 5 and 6, and the right part - for all values a different from 5. Do you think this equality will be an identity?

6.11 Simplification of expressions with algebraic fractions

So, now you know the formulas of abridged multiplication, you know how to factor polynomials and perform the operations with algebraic fractions. All this knowledge will allow you to solve various mathematical problems.

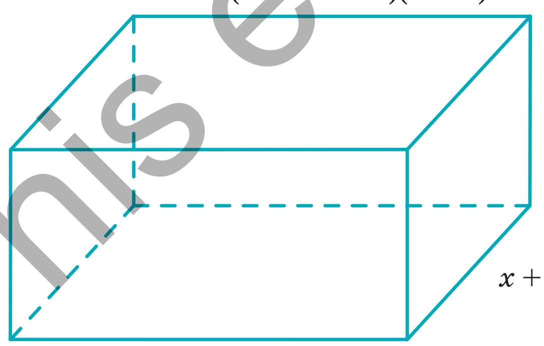
1. Use the polynomials $y^2 - \frac{9}{25}$ **and** $5y-3$ **and write:**

а) three integers;
б) three fractional expressions.

2. Work with a drawing. Find an unknown value.

a)

b)



3. Plot the graph of the function:

а) $y = \frac{x^2 - 6x + 9}{3 - x}$; б) $y = \frac{25x^2 - 16}{5x + 4}$;

в) $y = x - \frac{x-3}{3-x}$; в) $y = \frac{2x^3}{|x|}$.

4. Find the value of the expression, if given that a and b satisfy the condition $2a - 6b = 1$**:**

а) $\frac{1}{3b-a}$; б) $\frac{4a-12b}{9}$; в) $\frac{9b-3a}{7a-21b}$; г) $\frac{1}{a^2-6ab+9b^2}$; д) $\frac{4a^2-24ab+36b^2}{2a-6b-1}$.

5. Simplify the expression:

a) $\dfrac{\left(x^2-25\right)^2-25\left(x-5\right)^2}{25+x^2-10x}$; b) $\dfrac{5x+z-3y}{4x^2+9y^2+z^2-12xy+4xz-6yz}-\dfrac{1}{2x+y-3y}$;

c) $\left(\dfrac{2k\left(n+m\right)^3}{3\left(m-n\right)^2}\right)^4:\left(\dfrac{\left(m+n\right)^4}{3\left(m-n\right)}\right)^3+\left(-\dfrac{2}{m-n}\right)^5$.

6. Prove the identity:

a) $\dfrac{2x^3-6xy+2y^2}{2x^2+2y^2}+\dfrac{x^3+2xy-y^2}{x^2+y^2}=0$; b) $\dfrac{1}{(m+2)\left(m+3\right)}+\dfrac{1}{(m+3)\left(m+4\right)}=\dfrac{4}{m(m+4)}$.

7. The expressions describing the values given in the task below are presented on the board. Explain the meaning of each value. Use these expressions to make an equation to solve the problem.

Marat and Murat took part in a 100 km bicycle ride. Marat was riding at the speed of 5 km/h faster than Murat and arrived at the destination point 1 hour 45 minutes earlier. What is the speed of each cyclist?

$\dfrac{100}{x-5}$; $\dfrac{100}{y+1\frac{3}{4}}$; $\dfrac{100}{x-5}-\dfrac{100}{x}$; $\dfrac{100}{y}-\dfrac{100}{y+1\frac{3}{4}}$; $(x-5)\left(y+1\dfrac{3}{4}\right)$.

8. Find the values of a and b, at which the equality is correct:

a) $\dfrac{1}{(2x+5)(x+3)}=\dfrac{a}{2x+5}+\dfrac{b}{x+3}$; b) $\dfrac{5x+31}{(x-5)(x+2)}=\dfrac{a}{x-5}+\dfrac{b}{x+2}$.

6.12 What have I learned?

When you finish, you will repeat what you learned about algebraic fractions.

Algebraic fractions

Algebraic fractions.
An algebraic fraction is … .
A fraction is meaningful, if … .
A fraction is equal to zero, if … .
A tolerance range of a variable is … .

The basic property of algebraic fraction
If the numerator and denominator of an algebraic can be multiplied by or …

Operations with algebraic fractions
To add (subtract) two algebraic fractions with the same denominators, you have to…
To add (subtract) two algebraic fractions with different denominators, you have to…
To multiply one algebraic fraction by another, you have to … .
To find a quotient of two algebraic fractions, you have to … .
To raise an algebraic fraction to a power, you have to … .

Questions to help you repeat the previously studied materials
Write sentences using the following words at least once:
- an integer algebraic expression;
- a fractional algebraic expression;
- an algebraic fraction;
- tolerance range of a variable;
- the basic property of a fraction;
- reduction of fractions;
- addition and subtraction of algebraic fractions;
- multiplication and division of algebraic fractions;
- raising algebraic fractions to nth power.

1. Find the tolerance range of the variable x and represent it on the coordinate line:

a) $\frac{6}{x-5}$; b) $\frac{x-3}{x+8}$; c) $\frac{2+x}{6-x}$; d) $\frac{x-12}{x^2-9}$; e) $\frac{7}{(x-8)(x+1)}$; f) $\frac{7}{|x|+3}$.

2. Establish the conditions upon which the fraction takes even values:

a) $\frac{x^2-16}{x+4}$; b) $\frac{x^2-a^2}{x-a}$; c) $\frac{x^3-a^3}{x-a}$.

3. Is it true that:

a) two algebraic fractions can be equal with different numerators and same denominators;
b) two algebraic fractions can be equal with different numerators and denominators;
c) the sum of two algebraic fractions can be a monomial;
d) the difference between two algebraic fractions is an algebraic fraction;
e) the sum of two algebraic fractions can be equal to 0?

Explain your answer.

4. Is it possible that the fraction $\frac{a+8}{a+6}$, where a is an integer, takes integer values? Explain your answer.

5. We know that the expression $x+\frac{1}{x}$ takes integer values. Is it correct that the expressions $x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$ also take integer values?

6. Simplify the expression and find its value:

Value of a variable / Expression	$a=4$	$a=-2$
$\frac{1}{a^2-1}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-9}$		
$\dfrac{2-\frac{1}{a+1}+\frac{1}{a-1}}{a+\frac{a}{a^2-1}}$		

7. Prove the identity:

a) $\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right):\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right)=\frac{a^2+b^2}{ab}$;

b) $\frac{a^3+b^3}{a+b}-ab=(a-b)^2$;

c) $\left(\frac{m-1}{3m^2+6m+3}-\frac{1}{2m+2}\right):\frac{2m+10}{m^2+2m+1}=-\frac{1}{12}$;

d) $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}=\frac{1}{(c-a)(b-a)}$.

8. Translate the problem into mathematical language, and make a mathematical model:

a) Working together, Arman and Damira can complete the order and bake baursaks in a hours. Working alone, Arman could bake them in b hours. How long would it take Damira to complete the order?

b) The boat is sailing a h from point A to point B down stream, and b h from point B to point A. How many hours from A to B does the log float?

6.13 What do I know? Assessment activities

1. Write an algebraic fraction:

a) The numerator of which is equal to the sum of x and y, and the denominator - to the product of these numbers;
b) The numerator of which is equal to the cube of difference between a and b, and the denominator - to the sum of cubes of these numbers;
c) The numerator of which is equal to the difference of squares of m and n, and the denominator - to the sum of squares of these numbers.

2. What are the values of the variable x, at which the algebraic fraction is meaningless:

a) $\frac{5x}{x-5}$; b) $\frac{3a}{(a-2)(a+2)}$; c) $\frac{x-3}{x^2-9}$; d) $\frac{x-2}{|x|-2}$; e) $\frac{5x}{5x^2+19}$?

3. Fill the gaps, so the equality is correct:

a) $\frac{2a}{5b}=\frac{\dots}{25b}=\frac{8a^2}{\dots}$;

b) $\frac{x}{4y^2}=\frac{ax}{\dots}=\frac{\dots}{4b^2y^2}$;

c) $\frac{x+y}{x-y}=\frac{\dots}{x^2-y^2}=\frac{x^3+y^3}{\dots}$.

4. Represent the expression $3a-\frac{3a^2-7}{a}$ **in the form of an algebraic fraction. What is the minimum natural value of the obtained fraction?**

5. It is known that $\frac{a-3b}{b}=2$. **What is** $\frac{a}{b}$?

6. Reduce the fraction $\frac{ac+bx+ax+bc}{ay+3bx+3ax+by}$ **and find its value, if** $a=32\frac{3}{7}$, $b=16\frac{9}{11}$, $c=2{,}5$, $x=7{,}5$, $y=17{,}5$.

7. Simplify the expression:

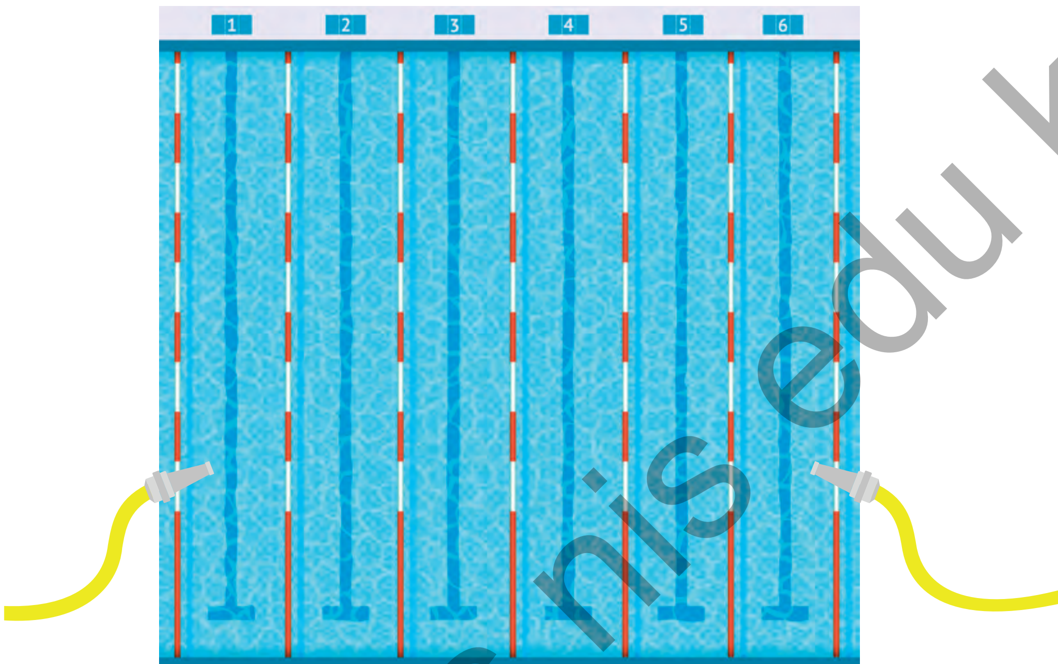
a) $\left(\frac{1}{a+2}+\frac{1}{a-2}-\frac{1}{a^2-4}\right)\cdot\frac{a^2-4a+4}{2a-1}$;

b) $\left(\frac{7}{8a^2-18b^2}-\frac{1}{2a^2+3ab}-\frac{1}{4ab-6b^2}\right):\left(\frac{a}{a-3b}+\frac{b}{a-b}\right)$.

8. Translate the problem into mathematical language, and make a mathematical model:

a) One of the numbers is m times less than the other. Find a larger number, if their arithmetic mean is n.

b) The swimming pool of a sports centre is filled through two pipes. The first pipe fills the swimming pool in a hours, and the second - in b hours. How long will it take to fill the swimming pool through both of pipes?



Review

1. Simplify the expression:

a) $0,63xy^3 \cdot 3\frac{1}{3}x^4y^2$;   b) $\left(2\frac{1}{2}a^2b^3\right)^5 \cdot \left(\frac{2^2\cdot 3}{5^2}\right)^2 a^8y^2$.

2. Find the value of the expression:

a) $(x^2+1)^2+(x-1)(x^2+1)-x^2$, if $x=-1$;

b) $(t+1)(t^2+1)(t-1)-(-1-t)^2$, if $t=-\frac{1}{2}$.

3. Calculate:

a) $(6^{30}+1)\left(6^{30}-1\right)-81^{15}\cdot 8^{20}$;   b) $5^{64}-(5^2-1)\left(5^2+1\right)\left(5^4+1\right)\left(5^8+1\right)\left(5^{16}+1\right)\left(5^{32}+1\right)$.

4. Represent as a polynomial, and write it in a standard form: $\overline{abc}+\overline{b2a}$.

5. Represent in the form of the product:

a) $3xy-2xy^2$;   b) $3x-3y+xz-yz$;

c) $x^2-4x+4-9y^2$;   d) a^4+4b^4.

6. Simplify the expression:

a) $\frac{x^2+1}{1-2x+x^2}+\frac{1+x}{x-1}$;   b) $a+5-\frac{10a}{a+5}$;

c) $\frac{x^2+2xy+y^2}{x^2+xy+6x+6y}:\frac{36-x^2}{x^2+xy-6x-6y}$;   d) $\left(t-\frac{2t-9}{t+8}\right)\cdot\left(\frac{t^2+3t}{t^2-64}\right)^{-1}+\frac{24}{t}$.

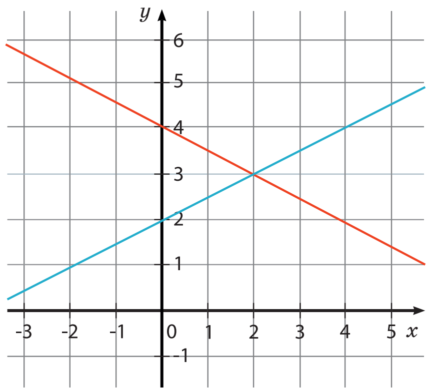
7. Prove that if $x+y+1=t$**, then** $tx+x+ty+y+1-t^2=0$.

8. Plot the graphs of the functions:

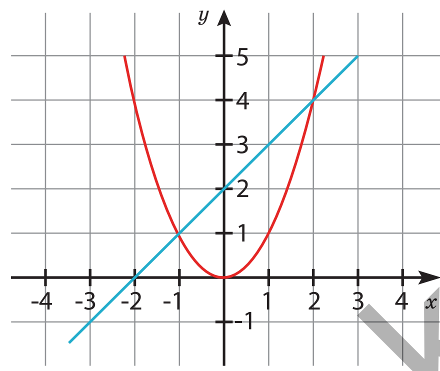
a) $y=8x^3-(2x+1)(4x^2+2x+1)$;   **b)** $y=(x+5)^2-(x+1)(x+4)$.

9. Write a system of equations with two variables, the graphs of which are given below:

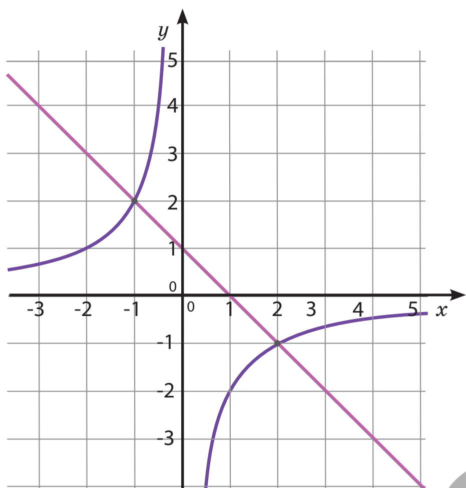
a)



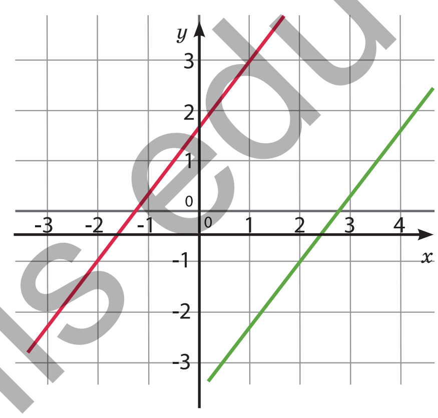
b)



c)



d)



10. Find the ordinate of the point of intersection between the graphs of the functions $y = px + 14$ **and** $y = (3 - p)x + p$ **, if the abscissa of the intersecting point equals 2.**

11. What is the value of *k*, so the graph of the function $y = kx + 3$ **passes through the point of intersection between the graphs of the functions** $y = -2x^2$ **and** $y = \frac{2}{x}$ **?**

12. What are the values of a parameter *p*, at which:

a) **the point of intersection between the graphs of the functions** $y = x + 3$ and $y = -4x - a + 4$ is in the 4^{th} quarter;

b) the graphs of the functions $y = (2p - 3)x + p + 6$ and $y = (4p - 1)x + 5 + 3p$ will be parallel?

13. Given the function $y = x - 2x + 3x - 5x + ... - 16x$ **. What is the coefficient k of this function?**

14. Alima wrote four consecutive even natural numbers, the sum of squares of which is 696. What numbers did Alima write?

15. The boat sailed for 2.4 hours downstream and 3.6 hours upstream. The distance travelled downstream is 5.4 km longer than the distance travelled upstream. What is the speed of the boat, if the speed of the stream is 2.5 km/h?

Review

1. Determine the type of a triangle, if one side of the triangle is 4 dm, the other - 30 cm and the perimeter is 0.11 m.

2. The length of the side of an isosceles triangle is 2 cm shorter than the base, and the perimeter of the triangle is 30 cm. What is the base of the triangle?

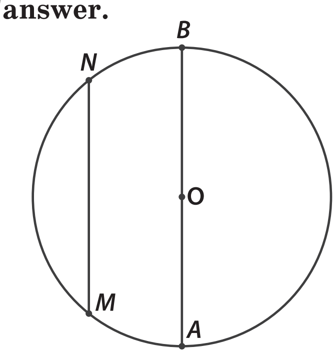
3. The angle A in a rectangular triangle ABC ($\angle C = 90°$) is 30°. The height CD is drawn from the vertex of the right angle. The length of the segment $BD = 2$ cm. What is the length of the segment AD?

4. Arman drew a triangle ABC. He drew a median and height from the vertex B of this triangle, so they divided the angle ABC into three equal angles. What are the angles of the triangle ABC?

5. Gaukhar drew a circle, drew a diameter AB and chords AC and BC, where $BC = AC$. Find the value of the angle AOS.

6. The diameter and chord that is equal to radius were drawn through the given point in a circle. What is the angle between diameter and chord? Explain your answer.

7. Gaukhar drew a circle and chord MN parallel to diameter AB. The figure is given below. It turned out to be that the distance between the chord and diameter is equal to half radius of the circle. Find the value of the angle between chord MA and AB.



8. How to divide a circle into 6 equal parts using a compass?

9. Arman has a triangle, one angle of which is 25^0. How to construct an angle equal to 125^0?

10. (An old problem) Three turtles are crawling across the desert. The first one says, "There is nobody ahead of me, and there are two turtles behind me". The second says, "There is one turtle ahead of me and one turtle behind me". The third says, "There is one turtle ahead of me and one turtle behind me". How is that possible?

11. Given a circle ω with a radius of 7 cm and line b. Determine the relative position of this line and circle, if the distance from the center of the circle to this line is d. Match the following:

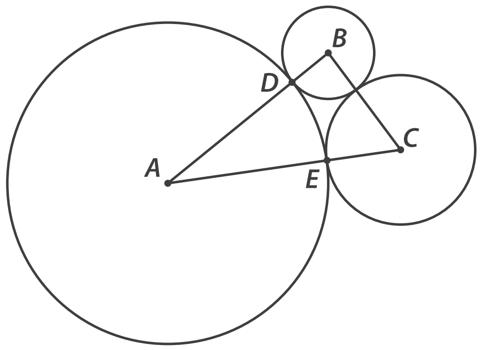
	Value of d	Relative position of circles
1	0	a) The line passes through the centre of the circle
2	3	b) The line and circle have two common points
3	7	c) The line is tangent to the circle
4	10	d) The line and circle do not have common points

12. Draw two circles with radii of 3 and 5 cm, so:

a) they do not have common points;
b) they are concentric;
c) they are externally tangent;
d) they are internally tangent.

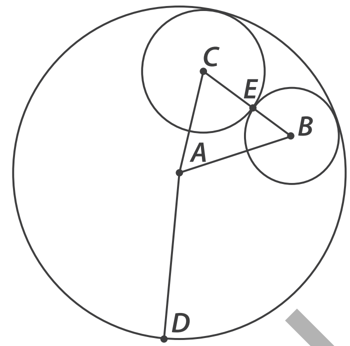
13. Solve the problems:

a)



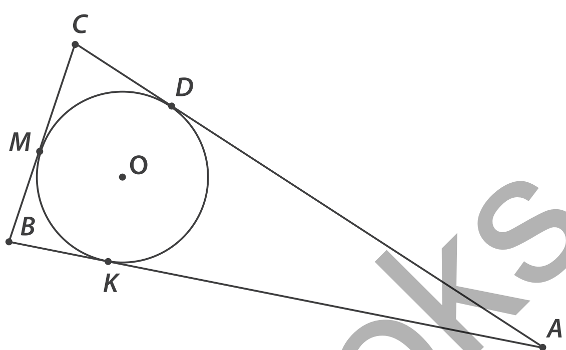
Given: Three externally tangent circles ω_1, ω_2, ω_3 with centres at points A, B and C, $AD = 7$, $BD = 2$, $CE = 4$.
Find: Perimeter of triangle ABC.

b)



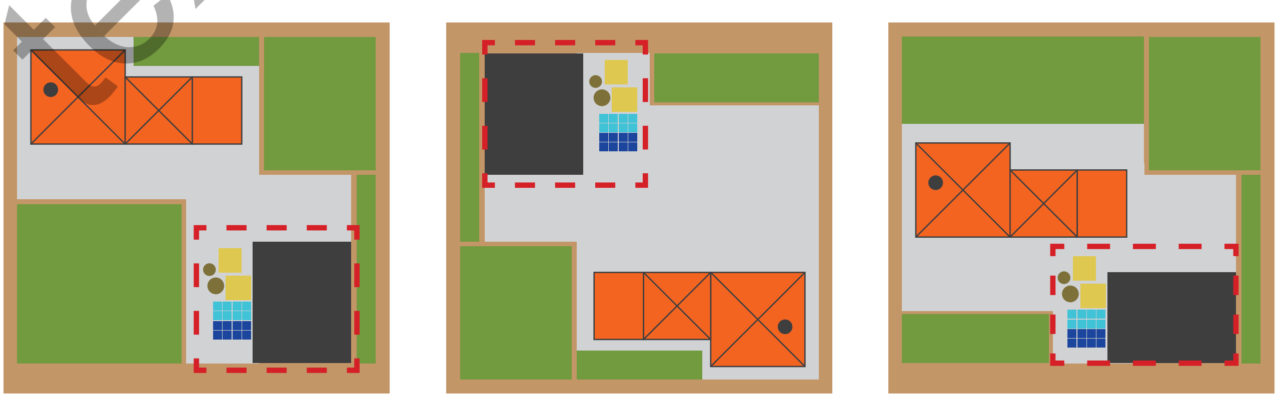
Given: Three internally tangent circles ω_1, ω_2, ω_3 with centres at points A, B and C, $AD = 7$, $BE = 2$, $CE = 4$.
Find: Perimeter of triangle ABC.

c) Given: $MC = 3$, $MB = 4$, $AK = 7$
Find: Perimeter of triangle ABC.



14. Given an incircle of a isosceles triangle. The points of tangency of the circle divide the side of the triangle in the ratio of 7:5 counting from the vertex. What are the sides of the triangle, if its perimeter is 144?

15. The owners of three houses decided to build a swimming pool. Where should it be placed, if it is known that the houses are not located in a line, and the distance from the swimming pool to the houses should be equal?



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